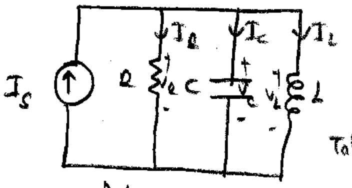
21.07.2010

$I_S = I_R + I_C + I_L$



$V_C = V_L = V_R$

$$I_S = \frac{V_C}{R} + C\frac{dV_C}{dt} + \frac{1}{L}\int_{t_0}^{t} V_C(t')dt' + I_0$$

(integro-diff equation)

Take derivative divide by c

$$\frac{1}{C}\frac{dI_S}{dt} = \frac{d^2V_C}{dt^2} + \frac{1}{RC}\frac{dV_C}{dt} + \frac{1}{LC}V_C$$

Zero-Input Response (Natural Response)

$I_S(t) = 0$ (response depend on the initial conditions)

$$\frac{d^2V_C}{dt^2} + \frac{1}{RC}\frac{dV_C}{Vt} + \frac{1}{LC}V_C = 0$$

$$V_C(t_0) = V_0 \qquad \frac{dV_C}{dt}(t_0) = -\frac{1}{RC}V_0 - \frac{I_0}{C}$$

$$\frac{dI_L}{dt} + \frac{1}{RC}\frac{dI_L}{dt} + \frac{1}{LC}I_L = 0$$

Same parameters similar solutions

$$I_L(t_0) = I_0 \qquad \frac{dI_L}{dt}(t_0) = \frac{V_0}{L}$$

Consider a general second order homogenous diff equation.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

$$x = e^{st}$$

$$\frac{d^2}{dt^2}(e^{st}) + a\frac{d}{dt}(e^{st}) + be^{st} = 0$$

$$s^2e^{st} + ase^{st} + be^{st} = 0$$

$$e^{st}\left[s^2 + as + b\right] = 0$$

$$s^2 + as + b = 0 \quad \text{(characteristic eqn.)}$$

$s_{1,2} = -\frac{a}{2} \mp \frac{\sqrt{a^2-4b}}{2}$ natural frequencies.

① if $s_1 \neq s_2$ $x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$A_1, A_2 \rightarrow$ coefficients to be determined due to initial conditions.

a) if $a^2 > 4b \Rightarrow s_{1,2} \Rightarrow$ real

$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ perfect exponential terms (increasing or decreasing)

b) if $a^2 < 4b$ $s_{1,2} \Rightarrow$ complex conjugate pairs

$s_1 = c + dj$ $s_2 = c - dj$

$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(c+dj)t} + A_2 e^{(c-dj)t}$

$= A_1 e^{ct} e^{djt} + A_2 e^{ct} e^{-djt}$

$= e^{ct}\left[A_1 e^{djt} + A_2 e^{-djt}\right]$ A_1 and A_2 are also complex conjugate pairs.

$= e^{ct}\left[(B_1 + B_2 j) e^{djt} + (B_1 - B_2 j) e^{-djt}\right]$ $A_1 = B_1 + B_2 j$ $A_2 = B_1 - B_2 j$

$= e^{ct}\left[B_1 (e^{djt} + e^{-djt}) + B_2 (j e^{djt} - j e^{-djt})\right]$

$= e^{ct}\left[2B_1 \frac{e^{djt} + e^{-djt}}{2} + B_2 (2j) \frac{e^{djt} - e^{-djt}}{2j} j\right]$

$= e^{ct}\left[2B_1 \cos(dt) - 2B_2 \sin(dt)\right]$

$= 2e^{ct}\left[B_1 \cos(dt) - B_2 \sin(dt)\right]$

$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \longrightarrow x(t) = e^{ct}\left[K_1 \cos(dt) + K_2 \sin(dt)\right]$

assume $s_1 \neq s_2$ *

$s_1 = c + jd$

$K_1 = 2B_1$,

$K_2 = -2B_2$,

(ii) If $s_1 = s_2 = \frac{-a}{2}$ (valid when $a^2 = 4b$)

$e^{s_0 t}$ is a solution $s_0 = s_{1,2} = \frac{-a}{2}$

and $te^{s_0 t}$ is also the solution of the diff-equation.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

$$\frac{d^2}{dt^2}(te^{s_0t}) + a\frac{d}{dt}(te^{s_0t}) + b(te^{s_0t}) \stackrel{?}{=} 0$$

$$\frac{d}{dt}\, te^{s_0t} = e^{s_0t} + s_0te^{s_0t}$$

$$\frac{d^2}{dt^2}(te^{s_0t}) = \frac{d}{dt}\left[e^{s_0t} + s_0te^{s_0t}\right]$$

$$= s_0e^{s_0t} + s_0e^{s_0t} + s_0^2te^{s_0t} = 2s_0e^{s_0t} + s_0^2te^{s_0t}$$

$$2s_0e^{s_0t} + s_0^2te^{s_0t} + a\left[e^{s_0t} + s_0te^{s_0t}\right] + bte^{s_0t} \stackrel{?}{=} 0$$

$$(s_0^2 + as_0 + b)te^{s_0t} + (2s_0 + a)e^{s_0t} \stackrel{?}{=} 0$$

$$s^2 + as + b = 0 \rightarrow \frac{d}{ds}(s^2 + as + b) = 2s + a\Big|_{s=s_0} = 0$$

$$(s - s_0)^2 = 0$$

$$\frac{d}{ds}(s-s_0)^2\Big|_{s=s_0} = 2(s-s_0)\Big|_{s=s_0} = 0$$

Hence $\Rightarrow$ $e^{s_0t} \rightarrow$ solution

$te^{s_0t} \rightarrow$ "

Thus, when $s_1 = s_2 = \frac{-a}{2} = s_0$

$$x(t) = (A_1 + A_2t)e^{s_0t} \quad \text{where } s_0 = -\frac{a}{2}$$

When $s_1 \neq s_2$;

* If $Re\{s_1\} > 0$ and/or $Re\{s_2\} > 0$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \lim_{x \to \infty} |x(t)| \to \infty$$

unbounded

* If $Re\{s_1\} \leq 0$ and $Re\{s_2\} \leq 0$

$$\lim_{t \to \infty} |x(t)| < M$$

bounded

When $s_1 = s_2 = s_0 = \frac{-a}{2}$

* If $s_0 \geq 0$ $(a \leq 0)$

$$x(t) = (A_1 + A_2)\, e^{s_0 t} \qquad \lim_{x \to \infty} |x(t)| \to \infty$$

unbounded

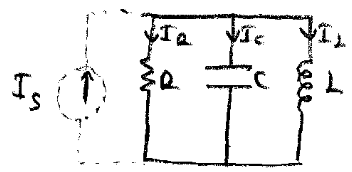
* If $s_0 < 0$ $(a > 0)$

$$x(t) = (A_1 + A_2 t)\, e^{s_0 t} \qquad \lim_{x \to \infty} |x(t)| \to 0$$

bounded

Back to zero-input solution of second order parallel circuit

$I_S(t) = 0$



$$\frac{d^2 V_C}{dt^2} + \frac{1}{RC}\frac{dV_C}{dt} + \frac{1}{LC} V_C = 0$$

$$V_C(t_0) = V_0 \qquad \frac{dV_C}{dt}(t_0) = -\frac{1}{RC} V_0 - \frac{1}{C} I_0$$

$$\frac{d^2 I_L}{dt^2} + \frac{1}{RC}\frac{dI_L}{dt} + \frac{1}{LC} I_L = 0$$

$$I_L(t_0) = I_0 \qquad \frac{dI_L}{dt}(t_0) = \frac{V_0}{L}$$

Let's define: $C>0, L>0, R>0$

$$2\alpha = \frac{1}{RC} \quad (\alpha \triangleq \text{damping constant (ratio)})$$

$$\alpha \geq 0$$

$$w_0^2 \triangleq \frac{1}{LC} \qquad w_0^2 > 0$$

$$w_0 > 0$$

resonant frequency

If $R \to \infty$ $\alpha = 0$ $\longleftrightarrow$ the circuit is lossless and passive (LC circuit)
(open circuit)

$$\frac{d^2V_c}{dt^2} + 2\alpha \frac{dV_c}{dt} + w_0^2 V_c = 0$$

$$V_c(t_0) = V_0 \qquad \frac{dV_c}{dt}(t_0) = -\frac{1}{RC}V_0 - \frac{1}{C}I_0 = -2\alpha V_0 - \frac{I_0}{C}$$

$$\frac{d^2I_L}{dt^2} + 2\alpha \frac{dI_L}{dt} + w_0^2 I_L = 0$$

$$I_L(t_0) = I_0 \qquad \frac{dI_L}{dt}(t_0) = \frac{V_0}{L},$$

$$s^2 + 2\alpha s + w_0^2 = 0$$

char-equation

$$S_{1,2} = -\alpha \mp \sqrt{\alpha^2 - w_0^2} \quad \text{natural-frequencies}$$

Let $t_0 = 0$

Let $-2\alpha V_0 - \frac{I_0}{C} = \gamma$

(i) $\alpha > w_o$ (overdamped case)

$$s_1 = -\alpha + \sqrt{\alpha^2 + w_o^2} < 0 \rightarrow \text{real}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - w_o^2} < 0 \rightarrow \text{negative}$$

$$\alpha_d \triangleq \sqrt{\alpha^2 - w_o^2}$$

$$s_{1,2} = -\alpha \pm \alpha_d$$

$$V_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \rightarrow \text{(Both exponentials are decaying)}$$

Let's determine A_1 and A_2:

$$\left.\begin{aligned} V_c(0) &= V_o = A_1 + A_2 \\ \left.\frac{dV_c}{dt}\right|_{t=0} &= s_1 A_1 + s_2 A_2 = \gamma \end{aligned}\right\} \text{Solve for } A_1 \text{ and } A_2$$

(ii) $\alpha = w_o$ (critically damped case)

$$s_{1,2} = s_o = -\alpha$$

$$V_c(t) = (A_1 + A_2 t) e^{-\alpha t}$$

$$V_c(0) = V_o = A_1$$

$$\left.\frac{dV_c}{dt}\right|_{t=0} = -\alpha A_1 + A_2 = \gamma$$

$$= -\alpha \underbrace{A_1}_{V_o} + A_2 = \gamma$$

$$A_2 = \gamma + \alpha V_o$$

(iii) $0<\alpha<\omega_0$ (underdamped).

$$\omega_d = \sqrt{\omega_0^2-\alpha^2} \longrightarrow \text{frequency of oscillation of damped sinusoidals.}$$

$$s_{1,2} = -\alpha \mp j\omega_d$$

$$Re\{s_1\} = Re\{s_2\} = -\alpha < 0$$

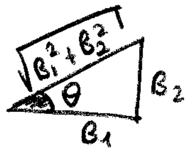
$$V_c(t) = A_1 e^{(-\alpha+j\omega_d)t} + A_2 e^{(-\alpha-j\omega_d)t}$$

$$= e^{-\alpha t}\left[A_1\cos(\omega_d t)+jA_1\sin(\omega_d t)+A_2\cos(\omega_d t)-jA_2\sin(\omega_d t)\right]$$

$$= e^{-\alpha t}\left[\underbrace{(A_1+A_2)}_{B_1\to real}\cos(\omega_d t)+\underbrace{j(A_1-A_2)}_{B_2\to real}\sin(\omega_d t)\right]$$

$$= e^{-\alpha t}\left[B_1\cos(\omega_d t)+B_2\sin(\omega_d t)\right]$$

$$= e^{-\alpha t}\sqrt{B_1^2+B_2^2}\left[\overbrace{\frac{B_1}{\sqrt{B_1^2+B_2^2}}}^{\cos\theta}\cos(\omega_d t)+\overbrace{\frac{B_2}{\sqrt{B_1^2+B_2^2}}}^{\sin\theta}\sin(\omega_d t)\right]$$



$$\cos\theta = \frac{B_1}{\sqrt{B_1^2+B_2^2}}$$

$$\sin\theta = \frac{B_2}{\sqrt{B_1^2+B_2^2}}$$

$$= \sqrt{B_1^2+B_2^2}\,e^{-\alpha t}\left[\cos[\omega_d t-\theta]\right] \quad \text{where } \boxed{\theta=\tan^{-1}\frac{B_2}{B_1}}$$

(iv) $\alpha = 0$ (lossless case) (LC circuit)

$s^2 + w_o^2 = 0$ char eq.

$s_{1,2} = \mp j w_o$

$Re\{s_1\} = Re\{s_2\} = 0$

$V_c(t) = A_1 e^{jw_o t} + A_2 e^{-jw_o t} \longrightarrow V_c(t) = B_1 \cos(w_o t) + B_2 \sin(w_o t)$

$= \sqrt{B_1^2 + B_2^2} \cos(w_o t - \theta)$

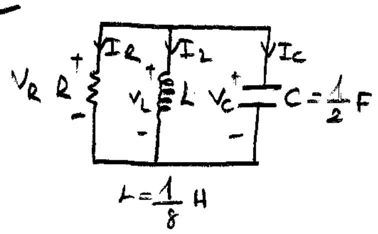
$\theta = \tan^{-1} \dfrac{B_2}{B_1}$

$V_c(0) = V_o = B_1$

$\dfrac{dV_c}{dt} = -w_o B_1 \sin(w_o t) + w_o B_2 \cos(w_o t)$

$\left.\dfrac{dV_c}{dt}\right|_{t=0} = w_o B_2 = \gamma \qquad B_2 = \dfrac{\gamma}{w_o}$

Ex.



$I_L(0) = -4$ Amper

$V_c(0) = 5$ Volt

Soltn.

$2\alpha = \dfrac{1}{RC} \longrightarrow \alpha = \dfrac{1}{R}$

$w_o^2 = \dfrac{1}{LC} = \dfrac{1}{\frac{1}{8}\,\frac{1}{2}} = 16$

$w_o = 4$ rad/sec

$\Rightarrow$

Find $V_C(t), I_L(t)$:

① $R = \frac{1}{5}\ \Omega$ $\quad \alpha = 5 \quad \omega_0 = 4$

$$S_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

$$= -5 \pm \sqrt{5^2 - 4^2} = -2, -8 \longrightarrow \text{real negative (overdamped)}$$

$$V_C(t) = A_1 e^{-8t} + A_2 e^{-2t}$$

$$V_C(0) = \boxed{5 = A_1 + A_2}$$

$$\left.\frac{dV_C}{dt}\right|_{t=0} = -8A_1 - 2A_2 = \frac{V_0}{RC} - \frac{I_0}{C}$$

$$= \frac{-5}{\frac{1}{2} \times \frac{1}{5}} - \frac{(-4)}{\frac{1}{2}} = -42$$

$$\boxed{-8A_1 - 2A_2 = -42}$$

$$A_2 = -\frac{1}{3}, \quad A_1 = \frac{16}{3}$$

$$\boxed{V_C(t) = -\frac{1}{3} e^{-2t} + \frac{16}{3} e^{-8t}}$$

$$I_L(t) = B_1 e^{-8t} + B_2 e^{-2t}$$

$$I_L(0) = \boxed{-4 = B_1 + B_2}$$

$$\left.\frac{dI_L}{dt}\right|_{t=0} = \frac{V_0}{L} = \frac{5}{\frac{1}{8}} = \boxed{40 = -8B_1 - 2B_2}$$

$$B_1 = -\frac{16}{3} \qquad B_2 = \frac{4}{3}$$

$$\boxed{I_L(t) = \frac{4}{3} e^{-2t} - \frac{16}{3} e^{-8t}}$$

OR

$$V_C = V_L = L \frac{dI_L}{dt}$$

$$V_C = L \frac{dI_L}{dt} = \frac{1}{8} \frac{d}{dt}\left[\frac{4}{3} e^{-2t} - \frac{16}{3} e^{-8t}\right]$$

$$= \frac{1}{8}\left[-\frac{8}{3} e^{-2t} + \frac{16 \times 8}{3} e^{-8t}\right]$$

$$= -\frac{1}{3} e^{-2t} + \frac{16}{3} e^{-8t}$$