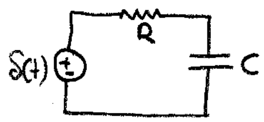
20.07.2010

Impulse Response



$V_c(0^-)=0$

$$\frac{dV_c}{dt}+\frac{1}{RC}V_c=\frac{1}{RC}\delta(t)$$

* Impulse response is denoted by $h(t)$.

when $t>0$ $\quad \frac{dh(t)}{dt}+\frac{1}{RC}h(t)=0$

$h(t)=K\,e^{-\frac{t}{RC}}u(t) \rightarrow$ put $h(t)$ in diff. eqn.

$$\frac{d}{dt}\left(Ke^{-\frac{t}{RC}}u(t)\right)+\frac{1}{RC}\left(K\,e^{-\frac{t}{RC}}u(t)\right)=\frac{1}{RC}\delta(t)$$

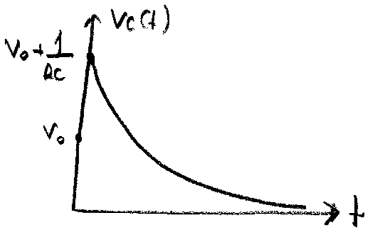
$$K\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}u(t)+Ke^{-\frac{t}{RC}}\delta(t)+\frac{K}{RC}\,e^{-\frac{t}{RC}}u(t)=\frac{1}{RC}\delta(t)$$

$$K\,e^{-\frac{0}{RC}}\delta(t)=\frac{1}{RC}\delta(t)$$

$$K\delta(t)=\frac{1}{RC}\delta(t) \quad \boxed{K=\frac{1}{RC}} \quad \boxed{h(t)=\frac{1}{RC}\,e^{-\frac{t}{RC}}u(t)}$$

* If $V_c(0^-)=V_0$

$$\underset{\text{input}}{\delta(t)} \longrightarrow V_c(t)=\left(V_0+\frac{1}{RC}\right)e^{-\frac{t}{RC}}u(t)$$



* The zero state impulse response is the time-derivative of zero state unit step response.

unit step response

$$g(t)_{\text{zero state}} = \left(1 - e^{-\frac{t}{RC}}\right)u(t)$$

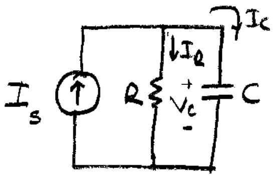
$$\frac{d}{dt} g(t)_{\text{zero state}} \overset{?}{=} h(t)_{\text{zero state}}$$

$$\frac{d}{dt}\left[\left(1-e^{-\frac{t}{RC}}\right)u(t)\right] = -\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}u(t) + \left(1-e^{-\frac{t}{RC}}\right)\delta(t)$$

$$= \frac{1}{RC} e^{-\frac{t}{RC}}u(t) + \underbrace{\left(1-e^{-\frac{0}{RC}}\right)}_{0}\delta(t)$$

$$= h(t)_{\text{zero state}} = \frac{1}{RC} e^{-\frac{t}{RC}}u(t)$$

Parallel RC Circuit



$V_c(0) = V_0$

$I_s = I_R + I_c$

$$I_s = \frac{V_c}{R} + C\frac{dV_c}{dt}$$

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{I_s}{C}$$

$V_c(0) = V_0$

Let $I_s(t) = \mu u(t)$

$t > 0$

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{\mu}{C}, \quad V_c(0) = V_0$$

① $V_{cp} = k$

$$\underbrace{\frac{d}{dt}(k)}_{0} + \frac{1}{RC}k = \frac{\mu}{C} \qquad k = \mu R$$

② $V_{ch} = k_1 e^{-\frac{t}{RC}}$

③ $V_c = V_{ch} + V_{cp}$

$V_c(t) = k_1 e^{-\frac{t}{RC}} + \mu R$

④ $V_c(0) = V_0 = k_1 + \mu R$

$k_1 = V_0 - \mu R$

$$V_c(t) = (V_0 - \mu R)e^{-\frac{t}{RC}} + \mu R$$

$\Longrightarrow$

Let $I_s(t) = M u(t)$ where $M = 1$

$$V_c(t) = (V_0 - R)\, e^{-\frac{t}{RC}} + R$$

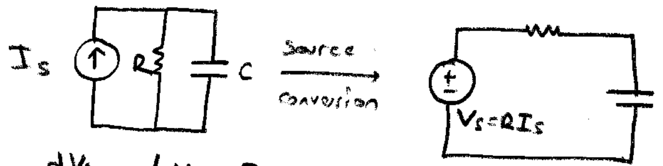
if $V_0 = 0$ (circuit is zero-state)

$$= V_{c\,\text{zero state}} = R\left[1 - e^{-\frac{t}{RC}}\right]$$

zero state response for unit step

$$h(t) = \frac{dV_{c(\text{zero state})}}{dt} = \frac{1}{C}\, e^{-\frac{t}{RC}}$$

zero state impulse response



$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{I_s}{C}$$

$$g(t) = \left(1 - e^{-\frac{t}{RC}}\right)$$

$$h(t) = \frac{1}{RC}\, e^{-\frac{t}{RC}}$$

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{V_s}{RC}$$

$$V_s = R I_s$$

$$g(t) = R\left(1 - e^{-\frac{t}{RC}}\right)$$

$$h(t) = \frac{1}{C}\, e^{-\frac{t}{RC}}$$

Linearity

input	output
$x(t)$	$\longrightarrow y(t)$
$kx(t)$	$\longrightarrow ky(t)$

if the condition is valid for all k values then the system is linear.

* The zero-state response of the LTI first order circuit is a linear function of the input.

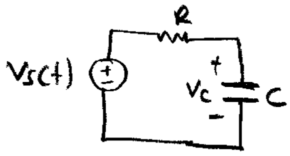
Time Inverience

input	output
$x(t)$	$\longrightarrow y(t)$
$x(t-t_0)$	$\longrightarrow y(t-t_0)$

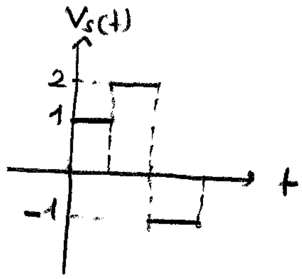
If the condition is satisfied for all t_0 values, then that system is time-invarient.

* The zero-state response of the LTI first order circuit is time-invariant.

Ex.



$V_c(0)=0$



Find $V_c(t) = ?$

Soltn.

If input $V_s(t)=u(t) \longrightarrow V_c(t) = \left(1 - e^{-\frac{t}{RC}}\right)u(t)$

However;

$$V_s(t) = u(t) + u(t-1) - 3u(t-2) + u(t-3)$$

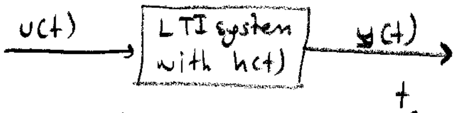
using linearity and time inverience

$$V_c(t) = \left(1 - e^{-\frac{t}{RC}}\right)u(t) + \left(1 - e^{-\frac{(t-1)}{RC}}\right)u(t-1) + -3\left(1 - e^{\frac{-(t-2)}{RC}}\right)u(t-2) + \left(1 - e^{-\frac{(t-3)}{RC}}\right)u(t-3)$$

if $V_c(0) = V_0$; there will be an extra term

$$V_0 e^{\frac{-t}{RC}} u(t) + \ldots$$

Convolution Integral

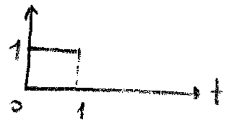


$$y(t) = u(t) \underset{\text{convolution}}{*} h(t) = \int_{\cancel{-\infty}\ t_0}^{t} v(t')h(t-t')dt' \quad \text{for } t \geq t_0 \quad y(t_0^-) = 0$$

Ex.

Find the zero-state response "$V_C(t)$" of series RC circuit for the following input;

$$v_s(t) = u(t) - u(t-1)$$



$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

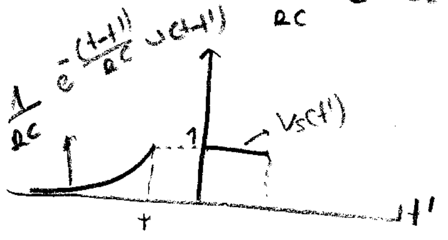
for $t \geq 0$

$$V_C(t) = \int_{-\infty}^{t} h(t-t')v_s(t')dt'$$

$$= \int_{-\infty}^{t} V_s(t-t')h(t')dt'$$

$$V_s(t') = \begin{cases} 1 & 0 \leq t' < 1 \\ 0 & \text{otherwise} \end{cases}$$

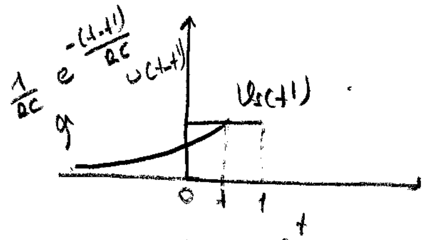
$$h(t-t') = \frac{1}{RC} e^{-\frac{(t-t')}{RC}} u(t-t')$$



$\Rightarrow$

$$V_C(t) = \int_{-\infty}^{t} h(t-t')v_s(t')dt' = 0$$

$0 \leq t < 1$



$$V_c(t) = \int_{-\infty}^{t} h(t-t')\, v_s(t')\, dt'$$

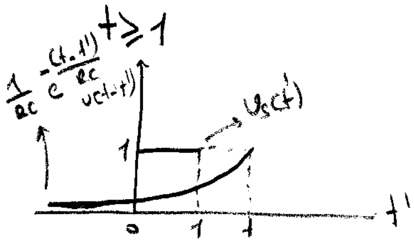
$$= \int_{-\infty}^{0} 0\, dt + \int_{0}^{t} 1\, \frac{1}{RC}\, e^{-\frac{(t-t')}{RC}}\, dt'$$

$$= \frac{1}{RC}\, e^{-\frac{t}{RC}} \int_{0}^{t} e^{\frac{t'}{RC}}\, dt'$$

$$= \left(\frac{1}{RC}\, e^{-\frac{t}{RC}}\right) RC\, e^{\frac{t'}{RC}} \Big|_{0}^{t}$$

$$V_c(t) = e^{-\frac{t}{RC}}\left[e^{\frac{t}{RC}} - 1\right] = 1 - e^{-\frac{t}{RC}}$$

$t \geq 1$



$$v_c(t) = \int_{-\infty}^{t} h(t-t')\, v_s(t')\, dt'$$

$$= \int_{-\infty}^{0} 0\, dt' + \int_{0}^{1} 1\left(\frac{1}{RC}\right) e^{-\frac{(t-t')}{RC}}\, dt' + \int_{1}^{t} 0\, dt'$$

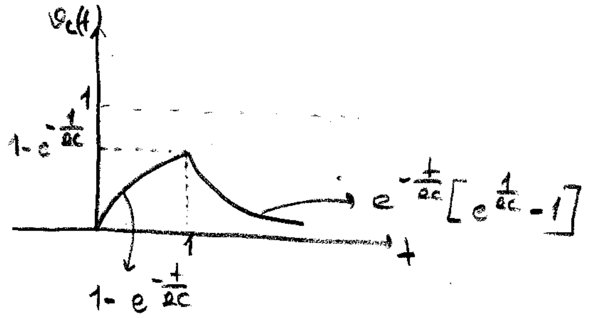
$$= \frac{1}{RC}\, e^{-\frac{t}{RC}} \int_{0}^{1} e^{\frac{t'}{RC}}\, dt'$$

$$= \left(\frac{1}{RC}\, e^{-\frac{t}{RC}}\right) RC\, e^{\frac{t'}{RC}} \Big|_{0}^{1}$$

$$= e^{-\frac{t}{RC}}\left[e^{\frac{1}{RC}} - 1\right]$$

General solution

$$v_c(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{RC}} & 0 \leq t < 1 \\ \left(e^{\frac{1}{RC}} - 1\right) e^{-\frac{t}{RC}} & t \geq 1 \end{cases}$$



Initial and Final Conditions

If unit step input is applied.

$V_C(0)$, $I_L(0)$ → initial conditions

$V_C(\infty)$, $I_L(\infty)$ → final conditions (steady-state values)

$$V_C(t) = \left[V_C(0) - V_C(\infty)\right] e^{-\frac{t}{\tau}} + V_C(\infty)$$

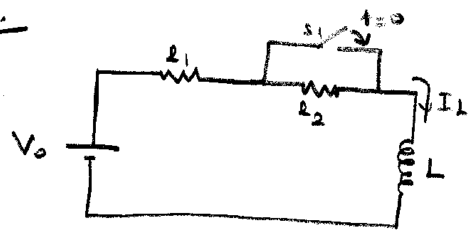
$$I_L(t) = \left[I_L(0) - I_L(\infty)\right] e^{-\frac{t}{\tau}} + I_L(\infty)$$

RC circuit $\tau = RC$

RL circuit $\tau = LG = \frac{L}{R}$

Time Constants

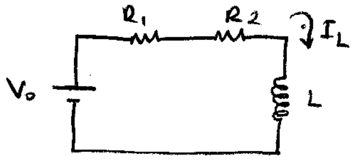
Ex.



Assume switch s_1 was open for a long time ($-\infty < t < 0$) and it is closed when $t=0$.

Find $I_L(t)$ $t>0$

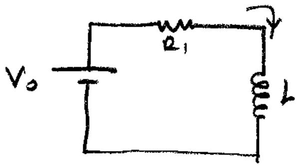
$-\infty < t < 0$



Since long time has passed $(-\infty<t<0)$, L will be have as short circuit;

$$I_L(0) = \frac{V_0}{R_1+R_2}$$

when $t>0$



as $t \to \infty$

L will once gain behave as short-circuit.

$$I_L(\infty) = \frac{V_0}{R_1}$$

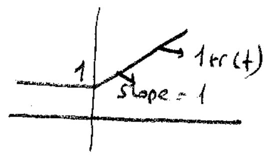
$$I_L(t) = (I_L(0) - I_L(\infty))e^{-\frac{t}{\tau}} + I_L(\infty)$$

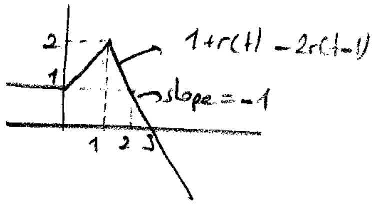
$$I_L(t) = \left(\frac{V_0}{R_1+R_2} - \frac{V_0}{R_1}\right) e^{-\frac{t}{\tau}} + \frac{V_0}{R_1} \qquad \boxed{\tau = \frac{L}{R_1}}$$

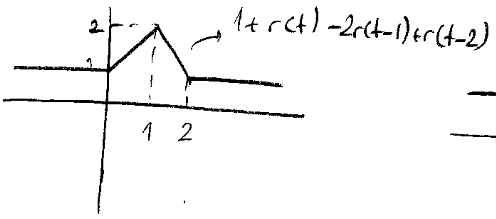
Midterm Konuları Sonu!!!

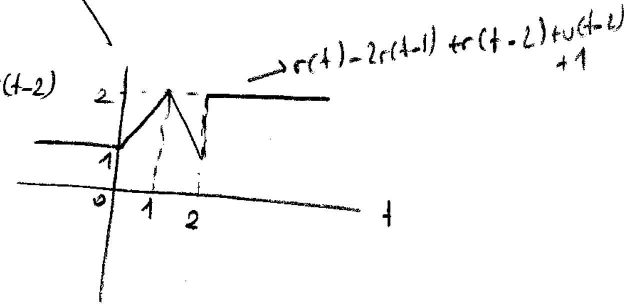
Ex.

$$x(t) = r(t) - 2r(t-1) + r(t-2) + u(t-2) + 1$$



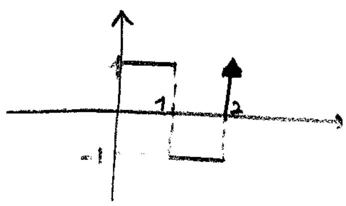






Let's draw;

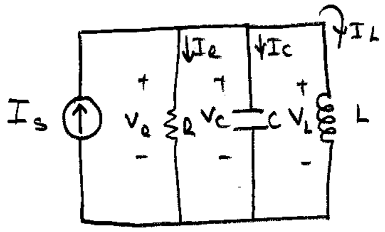
$$\frac{dx(t)}{dt} = u(t) - 2u(t-1) + u(t-2) + \delta(t-2) + 0$$



Second Order Circuits

It includes at least 2 energy storage elements that cannot be decomposed into a single circuit element.

Parallel LTI RLC Circuit



$V_C(t_0) = V_0$, $I_L(t_0) = I_0$

state variables I_L , V_C

$V_C = V_L = V_R$

$I_S = I_R + I_C + I_L$

$I_S = \frac{V_R}{R} + C\frac{dV_C}{dt} + I_L$

$I_S = \frac{V_C}{R} + C\frac{dV_C}{dt} + I_L$

① $\frac{dV_C}{dt} = \frac{1}{C} I_s - \frac{1}{CR} V_C - \frac{1}{C} I_L$

$V_L = V_C = L\frac{dI_L}{dt}$

② $\frac{dI_L}{dt} = \frac{V_C}{L}$

$$\frac{d}{dt}\begin{bmatrix} V_C \\ I_L \end{bmatrix} = \begin{bmatrix} \frac{1}{CR} & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_C \\ I_L \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} I_S$$

state-space representation.

Take derivative of ① put ② inside

$$\frac{d^2V_C}{dt^2} = \frac{1}{C}\frac{dI_s}{dt} - \frac{1}{CR}\frac{dV_C}{dt} - \frac{1}{C}\frac{dI_L}{dt}$$

$$\frac{d^2V_C}{dt^2} = \frac{1}{C}\frac{dI_s}{dt} - \frac{1}{RC}\frac{dV_C}{dt} - \frac{1}{LC}V_L$$

$$\frac{d^2V_C}{dt^2} + \frac{1}{RC}\frac{dV_C}{dt} + \frac{1}{LC}V_C = \frac{1}{C}\frac{dI_s}{dt}$$

$$V_c(t_0) = V_0$$

$$\frac{dV_c}{dt}(t_0) = \frac{I_s(t_0)}{C} - \frac{V_c(t_0)}{CR} - \frac{I_L(t_0)}{C}$$

$$\frac{dV_c}{dt}(t_0) = \frac{I_s(t_0)}{C} - \frac{V_0}{RC} - \frac{I_0}{C}$$

$$\frac{d^2 I_L}{dt^2} = \frac{1}{L}\frac{dV_c}{dt}$$

$$= \frac{1}{L}\left[\frac{1}{C} I_s - \frac{1}{CR} V_c - \frac{1}{C} I_L\right]$$

$$= \frac{1}{LC} I_s - \frac{1}{LCR} \underset{\overset{\downarrow}{L\frac{dI_L}{dt}}}{V_c} - \frac{1}{LC} I_L$$

$$\frac{d^2 I_L}{dt^2} = \frac{I_s}{LC} - \frac{1}{CR}\frac{dI_c}{dt} - \frac{1}{LC} I_L$$

$$\frac{d^2 I_L}{dt^2} + \frac{1}{RC}\frac{dI_L}{dt} + \frac{1}{LC} I_L = \frac{1}{LC} I_S$$

$$I_L(t_0) = I_0 \qquad \frac{dI_L}{dt}(t_0) = \frac{V_c(t_0)}{L} = \frac{V_0}{L}$$