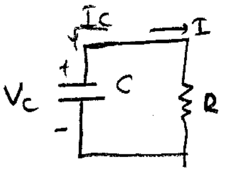
07.07.2010

## Zero-Input Response ($V_s=0, I_s=0$)



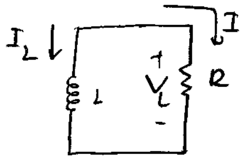

$V_c(0)=V_0$

$I_c=-I$

$C\frac{dV_c}{dt}=-\frac{V_c}{R}$

$$\boxed{\begin{gathered}\frac{dV_c}{dt}+\frac{1}{RC}V_c=0\\ V_c(0)=V_0\end{gathered}}$$

——— RC ———




$R=\frac{1}{G}$

$I_L=-I$

$I_L=-\frac{V_L}{R}$

$I_L=\frac{-L}{R}\frac{dI_L}{dt}$

$\frac{dI_L}{dt}+\frac{R}{L}I_L=0$

$I_L(0)=I_0$

——— RL ———

$$\boxed{\begin{gathered}\frac{dI_L}{dt}+\frac{1}{LG}I_L=0\\ I_L(0)=I_0\end{gathered}}$$

First order linear homogenous constant coefficient diff-eqv.

homogenous → RHS (The effect of input is zero)

| | |
|---|---|
| $V_c(t)=Ke^{st}$ | $I_L(t)=Me^{st}$ |

* K and M are determined due to initial conditions.

Put the solution in diff. eqn.

| | |
|---|---|
| $\frac{d}{dt}(Ke^{st})+\frac{1}{RC}(Ke^{st})=0$ | $\frac{d}{dt}(Me^{st})+\frac{1}{LG}(Me^{st})=0$ |
| $Kse^{st}+\frac{1}{RC}Ke^{st}=0$ | $Ms\,e^{st}+\frac{1}{LG}(Me^{st})=0$ |
| $\left(s+\frac{1}{RC}\right)\underbrace{Ke^{st}}_{\text{non zero}}=0$ | $\left(s+\frac{1}{LG}\right)\underbrace{Me^{st}}_{\text{non zero}}=0$ |
| $s=-\frac{1}{RC}$ (natural frequency) | $s=-\frac{1}{LG}$ (natural frequency) |
| $V_c(t)=Ke^{-\frac{t}{RC}}\quad t\geq 0$ | $I_L(t)=Me^{-\frac{t}{LG}}\quad t\geq 0$ |
| $V_c(0)=V_0=Ke^{-\frac{0}{RC}}$ | $I_L(0)=I_0=Me^{-\frac{0}{LG}}$ |
| $K=V_0$ | $M=I_0$ |
| $V_c(t)=V_0\,e^{-\frac{t}{RC}}$ | $I_L(t)=I_0\,e^{-\frac{t}{LG}}$ |
| $\tau=RC$ | $\tau=LG$ |

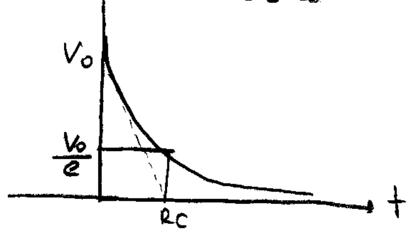
Time constants.

* Time constant: The time required for a zero input first order response to reach "$\frac{1}{e}$" of its initial condition value.

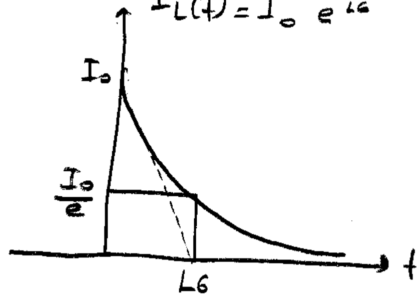
if $\tau$ is high $\longrightarrow$ slow system

" $\tau$ is low $\longrightarrow$ rapid system

$$V_c(t) = V_o e^{-\frac{t}{RC}}$$




$$I_L(t) = I_o e^{-\frac{t}{LG}}$$

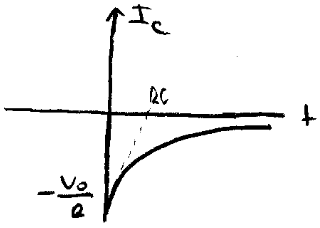



For practical application when $t = t \geqslant 5\tau$ we assume that $V_c(t) \cong 0$ $I_L(t) \cong 0$

$$I_c = C \frac{dV_c}{dt}$$

$$= C \frac{d}{dt} V_o e^{-\frac{t}{RC}}$$

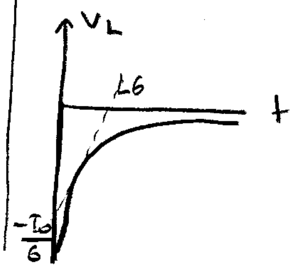
$$= \frac{-V_o}{R} e^{-\frac{t}{RC}}$$




$$V_L = L \frac{dI_L}{dt}$$

$$= L \frac{d}{dt} I_o e^{-\frac{t}{LG}}$$

$$= \frac{I_o}{G} e^{-\frac{t}{LG}}$$



## Complete Response

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} V_s(t)$$

$V_c(0) = V_0$

complete response $\rightarrow$ $V_c(t) = V_h(t) + V_p(t)$

$V_h$ = Homegenous (natural) solution.
(meaning solution when $V_s(t)=0$)
- It doesn't depend on input.
- It depends on initial condition, and circuit parameters.

$V_p$ = It is the solution due to input.
(forced response)

Step ① Find $V_p(t)$

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = V_s(t)$$

Generally, $V_p(t)$ is similar to $V_s(t)$.
- If $V_s(t)$ is sinusoidal, $V_p(t)$ is also sinusoidal.
- If " " DC, " " " DC.
- " " " exponential, " " " exponential.

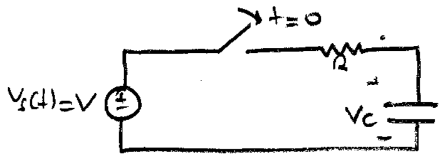
Step ② Determine $V_h(t)$

$$V_h(t) = K e^{-\frac{t}{RC}}$$

Step ③ $V_c = V_p + V_h$

Step ④ Use the initial condition $V_c(0) = V_0$ to determine $K$

## Constant Input ($V_s(t) = V$) constant




$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} V_s(t) = \frac{1}{RC} V$$

$$V_c(0) = V_0$$

Step ① $V_p = \mu \rightarrow$ constant

$$\underbrace{\frac{d}{dt}\mu}_{0} + \frac{1}{RC}\mu = \frac{1}{RC} V$$

$$\frac{1}{RC}\mu = \frac{1}{RC} V \qquad \mu = V = V_p$$

$$\boxed{V_p = V}$$

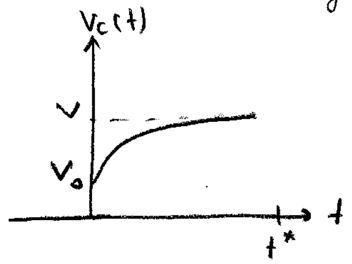
Step ② $V_h = K e^{-\frac{t}{RC}}$

Step ③ $V_c = V_p + V_h = V + K e^{-\frac{t}{RC}}$

Step ④ $V_c(0) = V_0 = V + K \underbrace{e^{-\frac{0}{RC}}}_{1}$

$$K = V_0 - V$$

Hence; $V_c(t) = \underbrace{(V_0 - V) e^{-\frac{t}{RC}}}_{\text{homogeneous}} + \underbrace{V}_{\text{forced}}$




$\tau = RC$

$t^* \geq 5RC$

$V_c(t) = \underbrace{(V_0 - V) e^{-\frac{t}{RC}}}_{\text{transient response}} + \underbrace{V}_{\text{steady-state response}}$

Steady state: The response observed when $t \to \infty$

transient response: The response that die out as $t \to \infty$

* Transient response is the result of sudden application of the input and the initial conditions.

Sinusoidal Input ($V_s = V_m \cos(\omega t)$)

$$\frac{d}{dt} V_c + \frac{1}{RC} V_c = \frac{1}{RC} V_s(t)$$

$$V_c(0) = V_0$$

Step ① $V_p = A_1 \cos(\omega t) + A_2 \sin(\omega t)$

$$\frac{d}{dt}(A_1 \cos(\omega t) + A_2 \sin(\omega t)) + \frac{1}{RC}(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = \frac{1}{RC} V_m \cos(\omega t)$$

$$-\omega A_1 \sin(\omega t) + \omega A_2 \cos(\omega t) + \frac{1}{RC}(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = \frac{1}{RC} V_m \cos(\omega t)$$

$$\left[\omega A_2 + \frac{A_1}{RC} - \frac{V_m}{RC}\right] \cos(\omega t) + \underbrace{\left[-\omega A_1 + \frac{A_2}{RC}\right]}_{0} \sin(\omega t) = 0$$

$t \geq 0$

$$A_1 = \frac{V_m}{1+(\omega RC)^2} \qquad A_2 = \frac{\omega RC V_m}{1+(\omega RC)^2}$$

$$V_p = \frac{V_m}{1+(\omega RC)^2} \cos(\omega t) + \frac{\omega RC V_m}{1+(\omega RC)^2} \sin(\omega t)$$

Step ② $V_h = K e^{-\frac{t}{RC}}$

Step ③ $V_c = V_p + V_h$

$$= K e^{-\frac{t}{RC}} + \frac{V_m}{1+(\omega RC)^2} \cos(\omega t) + \frac{\omega RC V_m}{1+(\omega RC)^2} \sin(\omega t)$$

Step ④ $V_c(0) = V_0 = K + \frac{V_m}{1+(\omega RC)^2} + 0$

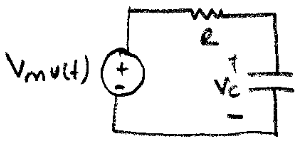
$$\boxed{K = V_0 - \frac{V_m}{1+(\omega RC)^2}}$$

$$V_c(t) = \underbrace{\left[V_o - \frac{V_m}{1+(\omega RC)^2}\right] e^{-\frac{t}{RC}}}_{\text{Homogeneous Transient part}} + \underbrace{\frac{V_m}{1+(\omega RC)^2}\left[\cos(\omega t) + \omega RC \sin(\omega t)\right]}_{\text{forced steady-state}}$$

$$V_L(t) = \left[V_o - \frac{V_m}{1+(\omega RC)^2}\right] e^{-\frac{t}{RC}} + \underbrace{\frac{V_m}{\sqrt{1+(\omega RC)^2}} \cos\left[\omega t - \tan^{-1}(\omega RC)\right]}$$

The forced response has a magnitude and phase shift composed to the input. However the frequency is same with the input.

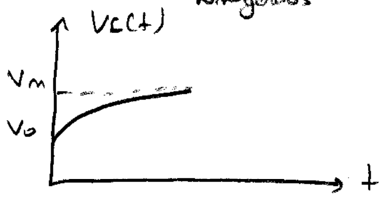
Unit Step Response = Constant Input Response
(Only first order RC or RL circuits)




$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{V_m u(t)}{RC}$$

$$V_c(0) = V_o$$

$$V_c(t) = \underbrace{(V_o - V_m) e^{-\frac{t}{RC}}}_{\text{homogeneous}} + \underbrace{V_m}_{\text{forced}} \quad t \geq 0$$




$$\lim_{t \to 0} V_c(t) = V_o$$

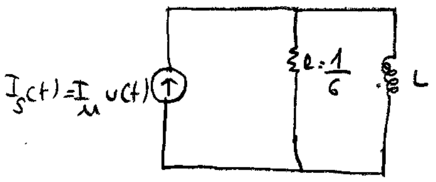
$$\lim_{t \to \infty} V_c(t) = V_m$$

$$V_c(t) = \underbrace{V_o e^{-\frac{t}{RC}}}_{\text{zero-input response}} + \underbrace{V_m (1 - e^{-\frac{t}{RC}})}_{\text{zero state response}}$$

zero-input response: $V_s(t) = 0$ — occurs when the input is 0.

zero state response: $V_c(0) = 0$ — occurs when the initial condition is 0 "$V_c = 0$"

## For RL Circuit




$$\frac{dI_L}{dt} + \frac{1}{LG} I_L = \frac{1}{LG} I_s(t) = \frac{1}{LG} I_M u(t)$$

$$I_L(0) = I_0$$

$$I_L(t) = \underbrace{(I_0 - I_M)\, e^{-\frac{t}{LG}}}_{\text{homogeneous (natural) response}} + \underbrace{I_M}_{\text{forced response}}$$

$$I_L(t) = \underbrace{I_0\, e^{-\frac{t}{LG}}}_{\text{zero-input response}} + \underbrace{I_M \left(1 - e^{-\frac{t}{LG}}\right)}_{\text{zero state response}}$$

One can think that ;

The RC and RL circuits have the INPUTS :

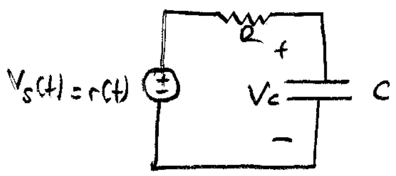
① The input that occured before $t=0$

$\Rightarrow$ zero-input response

② The input occured at $t=0$

$\Rightarrow$ zero-state response

## Ramp Response




$V_c(0) = V_0$

$r(t) = t\, u(t)$

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} r(t) = \frac{1}{RC} t\, u(t)$$

$t > 0$

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} t$$

$V_c(0) = V_0$

Step ① $V_p = A_1 t + A_2$

$$\frac{d}{dt}(A_1 t + A_2) + \frac{1}{RC}(A_1 t + A_2) = \frac{1}{RC} t$$

$$A_1 + \frac{1}{RC}(A_1 t + A_2) = \frac{1}{RC} t$$

$$\underbrace{\left(\frac{1}{RC} A_1 - \frac{1}{RC}\right)}_{0} t + \underbrace{\left(A_1 + \frac{A_2}{RC}\right)}_{0} = 0$$

$A_1 = 1$, $A_2 = -RC$

$$\boxed{V_p = t - RC}$$

Step ② $V_h = K e^{-\frac{t}{RC}}$

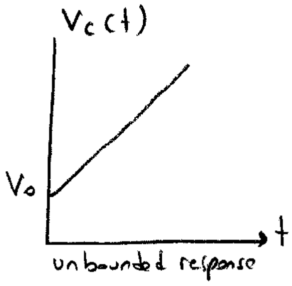
Step ③ $V_c = (t - RC) + K e^{-\frac{t}{RC}}$

Step ④ $V_c(0) = V_0$

$$V_0 = (0 - RC) + K$$

$$K = V_0 + RC$$

$$V_c(t) = (V_0 + RC) e^{-\frac{t}{RC}} + (t - RC)$$




$$\lim_{t \to \infty} V_c(t) = \infty$$

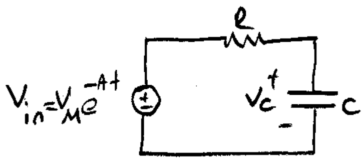
Note that, zero state unit step response is the time derivative of zero state unit ramp response.

$(V_0 = 0)$

$$V_{c_{\text{zero state ramp}}} = t - RC\left(1 - e^{-\frac{t}{RC}}\right)$$

$$\frac{d}{dt} V_{c_{\text{zero state ramp}}} = 1 - RC\left(0 - \frac{1}{RC} e^{-\frac{t}{RC}}\right) = \underbrace{1 - e^{-\frac{t}{RC}}}_{\text{zero state unit step response}}$$

## Exponential Response




$$\frac{dV_c}{dt}+\frac{1}{RC}V_c=\frac{1}{RC}V_s(t)$$

$$V_c(0)=V_0$$

a) $A \neq \frac{1}{RC}$

b) $A=\frac{1}{RC}$

a) Step ① $V_p = M e^{-At}$

$$\frac{d}{dt}M e^{-At}+\frac{1}{RC}M e^{-At}=\frac{1}{RC}V_m e^{-At}$$

$$-AM e^{-At}+\frac{M}{RC}e^{-At}=\frac{V_M}{RC}e^{-At}$$

$$\left(-A+\frac{1}{RC}\right)M=\frac{V_M}{RC}$$

$$M=\frac{V_M}{RC}\left(\frac{RC}{1-ARC}\right)$$

$$\boxed{M=\frac{V_M}{1-ARC}}$$

Step ② $V_h = k e^{-\frac{t}{RC}}$

Step ③ $V_c = V_p + V_h$

$$=\frac{V_M}{1-ARC}e^{-At}+k e^{-\frac{t}{RC}}$$

Step ④ $V_c(0)=V_0=\frac{V_M}{1-ARC}+k$

$$k=V_0-\frac{V_M}{1-ARC}$$

$$V_c(t)=\frac{V_M}{1-ARC}e^{-At}+\left(V_0-\frac{V_M}{1-ARC}\right)e^{-\frac{t}{RC}}$$

if $A>0$ both term are decaying.

if $A<0$ on term delays and the other terms is an increasing exponential.

ⓑ $A = \frac{1}{RC}$

Step ① $V_p = Mte^{-At}$

$= Mte^{-\frac{1}{RC}t}$

$$\frac{d}{dt}\left(Mte^{-\frac{1}{RC}t}\right) + \frac{1}{RC}\left(Mte^{-\frac{1}{RC}t}\right) = \frac{1}{RC} V_m e^{-At} = \frac{1}{RC} V_m e^{-\frac{1}{RC}t}$$

$$M\left[e^{-\frac{1}{RC}t} + t\left(-\frac{1}{RC} e^{-\frac{1}{RC}t}\right)\right] + \frac{1}{RC}\left(Mte^{-\frac{1}{RC}t}\right) = \frac{1}{RC} V_m e^{-\frac{1}{RC}t}$$

$$M e^{-\frac{1}{RC}t} \cancel{- Mt \frac{1}{RC} e^{-\frac{1}{RC}t}} \cancel{+ Mt \frac{1}{RC} e^{-\frac{1}{RC}t}} = \frac{1}{RC} V_m e^{-\frac{1}{RC}t}$$

$$M e^{-\frac{1}{RC}t} = \frac{1}{RC} V_m e^{-\frac{1}{RC}t}$$

$$M = \frac{V_m}{RC}$$

$$V_p(t) = \frac{V_m}{RC} t\, e^{-\frac{t}{RC}}$$

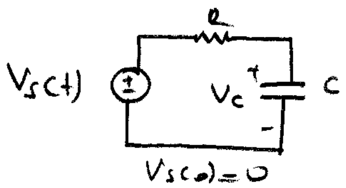
Step ② $V_h = K e^{-\frac{t}{RC}}$

Step ③ $V_c = V_h + V_p = \frac{V_m}{RC} t e^{-\frac{t}{RC}} + K e^{-\frac{t}{RC}}$

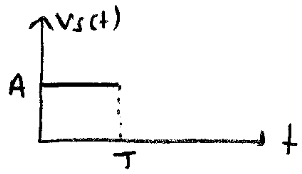
Step ④ $V_c(0) = V_0 = K$

$$V_c(t) = \frac{V_m}{RC} t e^{-\frac{t}{RC}} + V_0 e^{-\frac{t}{RC}}$$

<u>Pulse Response</u>




$V_s(t) = A[u(t) - u(t-T)]$



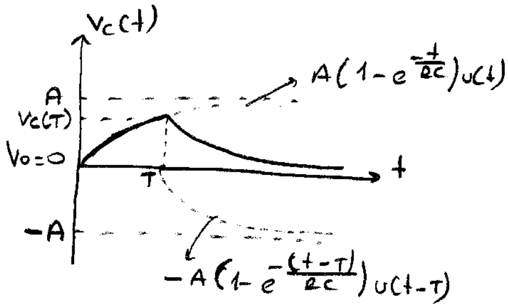

Since the initial condition is zero "$V_c(0) = 0$", the total response can be found as the addition (superposition), of the responses caused by step function applied at $t=0$ and step function applied when $t = T$.

$$V_s(t) = \underbrace{Au(t)}_{V_1(t)} \underbrace{-Au(t-T)}_{V_2(t)} = V_1(t) + V_2(t)$$

$$V_1(t) = Au(t) \longrightarrow A\left(1 - e^{-\frac{t}{RC}}\right) \quad (\text{zero state response } Au(t))$$

$$+ \quad V_2(t) = -Au(t-T) \longrightarrow -A\left(1 - e^{-\frac{(t-T)}{RC}}\right)u(t-T) \quad ( \text{''} \quad \text{''} \quad \text{''} \quad Au(t-T))$$

$$V_s(t) \longrightarrow A\left(1 - e^{-\frac{t}{RC}}\right)u(t) - A\left(1 - e^{-\frac{(t-T)}{RC}}\right)u(t-T)$$




$$V_s(t) = \begin{cases} A & 0 \le t < T \\ 0 & T < t \end{cases}$$

when $0 \le t \le T$;

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{A}{RC} \qquad V_c(0) = 0$$

$$V_c(t) = A\left(1 - e^{-\frac{t}{RC}}\right) \qquad 0 \le t \le T$$

$$V_c(0) = 0$$

$$V_c(T) = A\left(1 - e^{-\frac{T}{RC}}\right)$$

when $T < t$;

$$V_s(t) = 0 \text{ (in this region)}$$

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = 0 \qquad V_c(T) = A\left(1 - e^{-\frac{T}{RC}}\right)$$

$$V_p = 0$$

$$V_h = K e^{\frac{-t}{RC}}$$

$$= M e^{-\frac{(t-T)}{RC}}$$

$$= M e^{\frac{-t}{RC}} e^{\frac{T}{RC}}$$

$$= M e^{\frac{T}{RC}} e^{-\frac{t}{RC}}$$

$$M e^{\frac{T}{RC}} = K$$

$$V_c = V_p + V_n = 0 + \mu\, e^{-\frac{(t-T)}{RC}}$$

$$V_c(T) = A\left(1 - e^{-\frac{T}{RC}}\right) = \mu\, e^{-\frac{(T-T)}{RC}}$$

$$\mu = A\left(1 - e^{-\frac{T}{RC}}\right)$$

$$V_c(t) = \underbrace{A\left(1 - e^{\frac{T}{RC}}\right)}_{\mu}\, e^{\frac{(t-T)}{RC}}$$

$$V_c(t) = A\left(1 - e^{-\frac{T}{RC}}\right) e^{\frac{T}{RC}}\, e^{-\frac{t}{RC}}$$

$$= A\left(e^{\frac{T}{RC}} - 1\right) e^{-\frac{t}{RC}}$$

What happens if there is non-zero initial condition?

<u>1st Method;</u>

$$V_c(t) = V_{\text{due-to-inputs}} + V_{\text{due-to-initial condition}}$$