04.02.2010

Inductors in AC Circuits

$\phi = LI$

$\frac{d\phi}{dt} = L\frac{dI}{dt} = V \xrightarrow[\text{Transform}]{\text{Laplace}} L\,s\,I(s) = V(s) \qquad \frac{V(s)}{I(s)} = sL$

Faraday's Law

$s = j\omega$ Frequency Domain

$\frac{V(j\omega)}{I(j\omega)} = j\omega L = j2\pi f L$

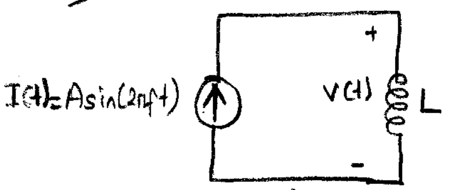
$X_L = 2\pi f L$

impedance of inductor.

* If f is increased $\left|\frac{V(j\omega)}{I(j\omega)}\right|$ increases assuming that $V(j\omega)$ is constant, that will cause $|I(j\omega)|$ to decrease.

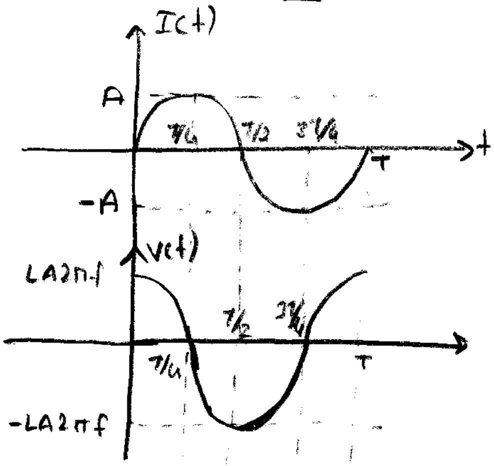
Hence, in AC excitation inductor behaves as open circuit.

Ex.



$V(t) = L\frac{dI(t)}{dt} = L\frac{d}{dt}\left[A\sin(2\pi f t)\right]$

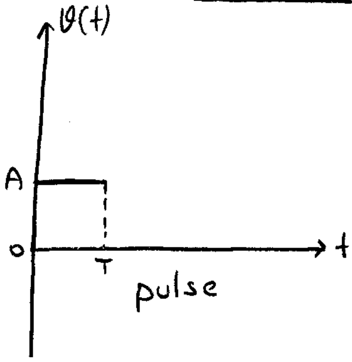
$V(t) = L A 2\pi f \cos(2\pi f t)$



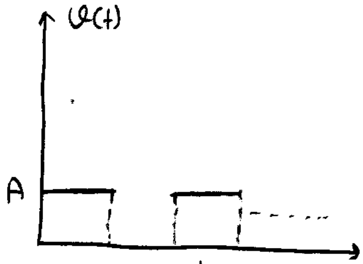
Current lags Voltage 90° for inductors

be behind

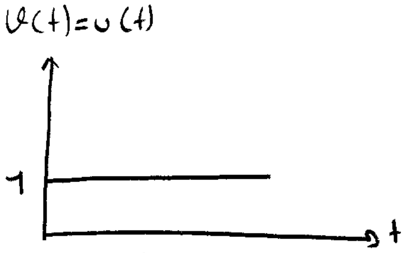
Typical Waveforms



pulse

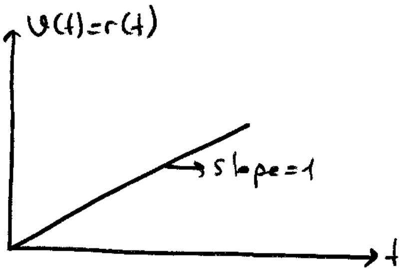


pulse train



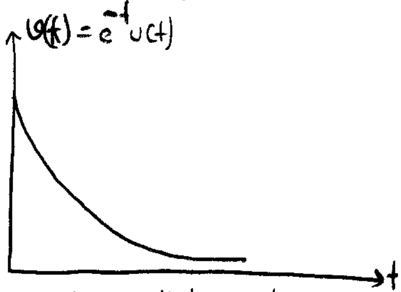
unit step

$$u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

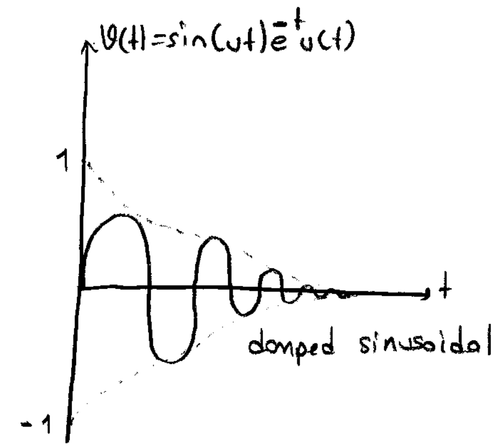


ramp

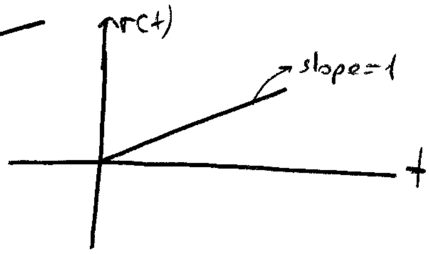
$$r(t) = \begin{cases} t & t\geqslant 0 \\ 0 & t<0 \end{cases}$$



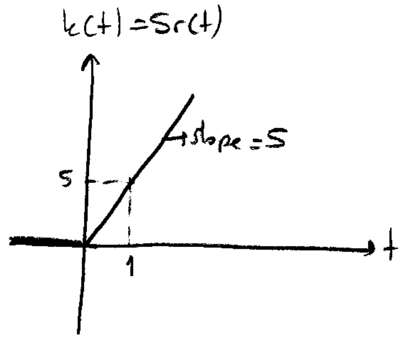
exponential signal

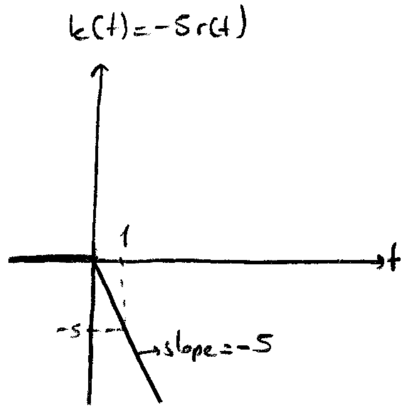


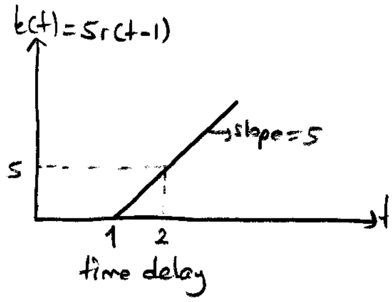
Ex.

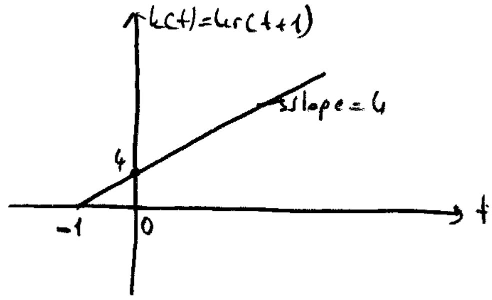


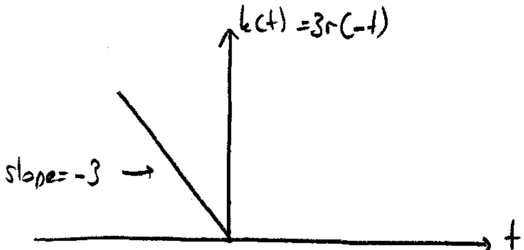
sketch $k(t)=5r(t)$ $k(t)=-3r(-t)$
$k(t)=-5r(t)$ $k(t)=-10r(-5t-3)$
$k(t)=5r(t-1)$
$k(t)=4r(t+1)$
$k(t)=3r(-t)$



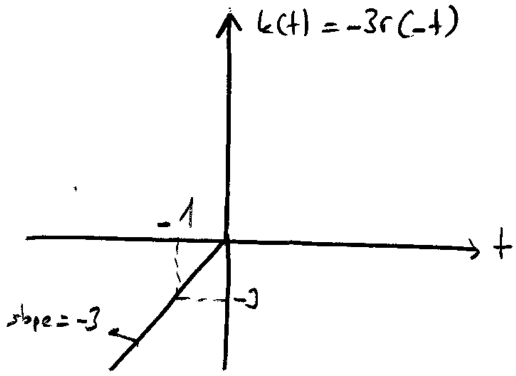




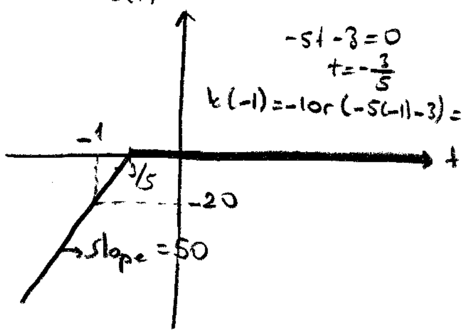




time inversion



$k(t) = -10r(-5t-3)$



$-5t-3=0$

$t=-\frac{3}{5}$

$k(-1) = -10r(-5(-1)-3) = -10r(5-3)$

$= -10r(2)$

$= -20$

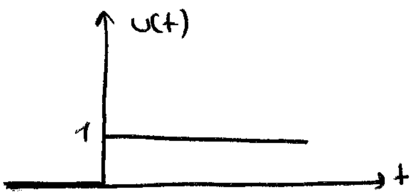
* $k(t) \longrightarrow k(at)$

  $|a|>1$ compresing

  $|a|<1$ exdending

Unit-step function

$$u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

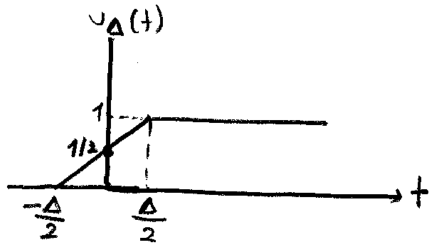


* u(t) has a jump discontinuity at t=0
* u(0) is generally undefined
* If it is necesorry to define a value for u(t) when t=0 $[0 \leq u(0) \leq 1]$

  for example= $u(0)=\frac{1}{2}$

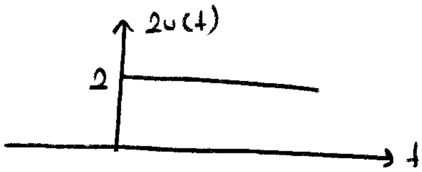
$u(t) = \frac{dr(t)}{dt}$ $\qquad \overset{\text{function}}{\widetilde{r(t)}} = \int_{-\infty}^{t} u(t')\,dt' \rightarrow$ indefinite integral $\qquad \begin{Bmatrix} 0 & k<0 \\ k & k>0 \end{Bmatrix} = \int_{-\infty}^{k} u(t')\,dt'$ (k constant)

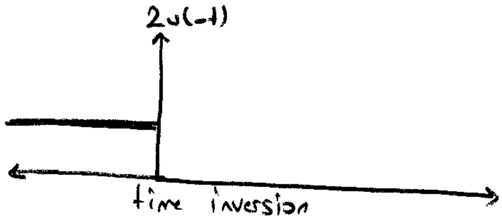
$\rightarrow$ definite integral



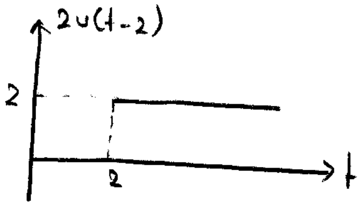
let $u(t) = \lim_{\Delta \to 0} u_\Delta(t)$

$u_\Delta(0) = \frac{1}{2} \rightarrow u(0) = \frac{1}{2}$

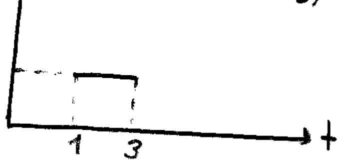




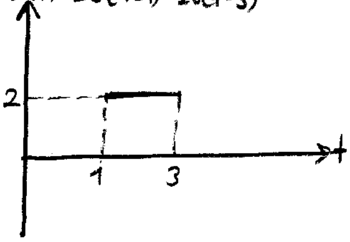
time inversion



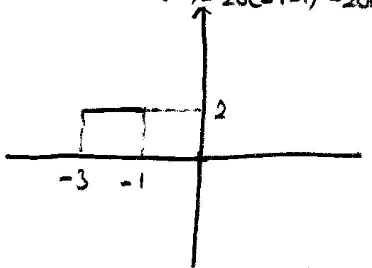
$u(t) = 2u(t-1) - 2u(t-3)$



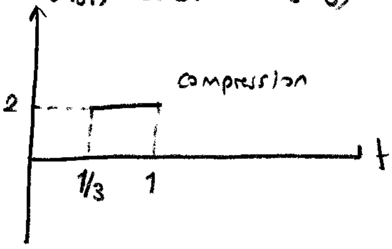
$u(t) = 2u(t-1) - 2u(t-3)$



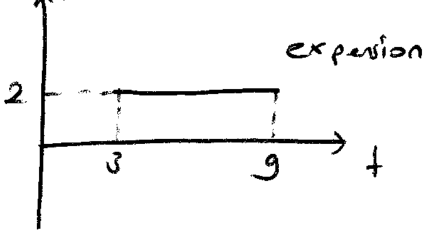
$u(-t) = 2u(-t-1) - 2u(-t-3)$



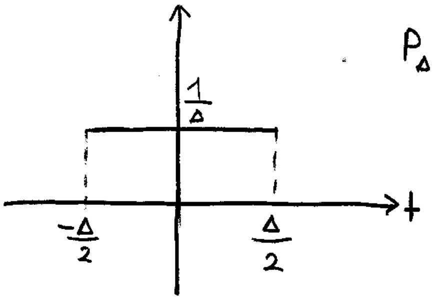
$u(3t) = 2u(3t-1) - 2u(3t-3)$



$u(\frac{1}{3}t) = 2u\left(\frac{1}{3}t - 1\right) - 2u\left(\frac{1}{3}t - 3\right)$



Pulse Function



$$P_\Delta(t) = \frac{d}{dt} u_\Delta(t)$$

$$P_\Delta(t) = \frac{u\left(t+\frac{\Delta}{2}\right) - u\left(t-\frac{\Delta}{2}\right)}{\Delta}$$

$$P_\Delta(t) = P_\Delta(-t)$$

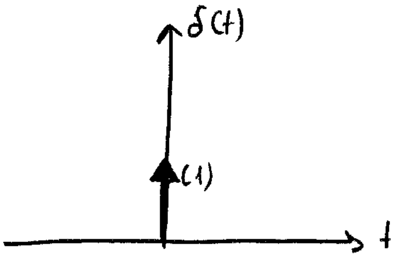
(even function)

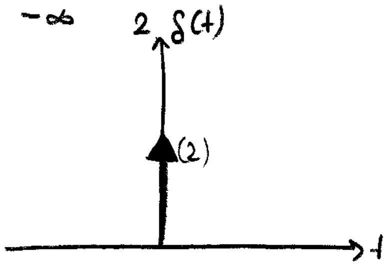
$$\int_{-\infty}^{\infty} P_\Delta(t)\,dt = 1$$

Impulse Function

$$\delta(t) = \lim_{\Delta \to 0} P_\Delta(t) = \frac{d}{dt} u(t)$$

$$\delta(t) = \frac{d}{dt} u(t) \longrightarrow u(t) = \int_{-\infty}^{t} \delta(t')\,dt'$$





* $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{singular} & t = 0 \end{cases}$

* $\delta(t) = \delta(-t)$ (Even function)

* $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

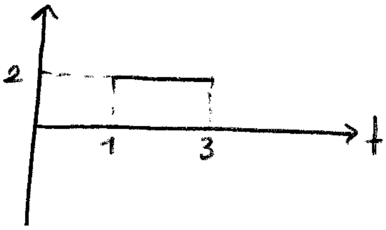
* If $f(t)$ is a continous function at $t=0$, then $f(t).\delta(t) = f(0).\delta(t)$

$$\int_{a}^{b} \delta(t)\,dt = \begin{cases} a<0,\ b<0 & \text{then } 0 \\ a<0,\ b>0 & \text{then } 1 \\ a>0,\ b>0 & \text{then } 0 \\ a>0,\ b<0 & \text{then } -1 \end{cases}$$

$$* \int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} f(0)\,\delta(t)\,dt = f(0)\underbrace{\int_{-\infty}^{\infty} \delta(t)\,dt}_{1} = f(0)$$

$$* \text{If } a \neq 0 \longrightarrow \delta(at) = \frac{1}{|a|}\,\delta(t)$$

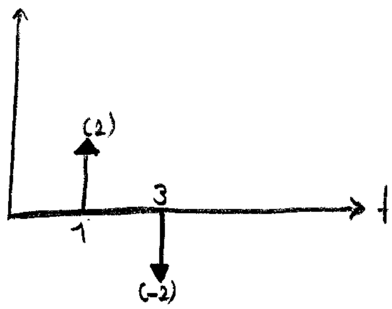
Ex. $v(t) = 2u(t-1) - 2u(t-3)$



Find $\dfrac{dv(t)}{dt} = ?$

Soln:

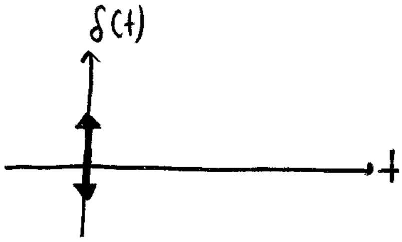
$$\frac{dv(t)}{dt} = 2\,\delta(t-1) - 2\,\delta(t-3)$$



Unit Doublet Function

$$\dot{\delta}(t) = \frac{d}{dt}\delta(t)$$

$$\int_{-\infty}^{t} \dot{\delta}(t')dt' = \delta(t)$$



* If $f(0)$ and $f'(0)$ are well-defined;

$$\frac{d}{dt}\left(f(t)\,\delta(t)\right) = f'(t)\,\delta(t) + f(t)\,\dot{\delta}(t)$$

$$\frac{d}{dt}\left(f(0)\,\delta(t)\right) = f'(0)\,\delta(t) + f(t)\,\dot{\delta}(t)$$

$$f(0)\frac{d}{dt}\delta(t) = f'(0)\,\delta(t) + f(t)\,\dot{\delta}(t)$$

$$f(0)\,\dot{\delta}(t) = f'(0)\,\delta(t) + f(t)\,\dot{\delta}(t)$$

$$\boxed{f(t)\,\dot{\delta}(t) = f(0)\,\dot{\delta}(t) - f'(0)\,\delta(t)}$$

* $$\int_{-\infty}^{\infty} f(t)\,\dot{\delta}(t)dt = f(t)\,\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)dt$$

$f = k \quad \dot{\delta}(t)dt = du$

$df = dk \quad \delta(t) = u$

$$\int_{-\infty}^{\infty} k\,du = ku\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u\,dk$$

$$= \underbrace{\left[f(\infty)\underbrace{\delta(\infty)}_{0} - f(-\infty)\underbrace{\delta(-\infty)}_{0}\right]}_{0} - \int_{-\infty}^{\infty} f'(0)\,\delta(t)\,dt$$

$$= 0 - f'(0)\int_{-\infty}^{\infty}\delta(t)dt$$

$$= -f'(0)$$

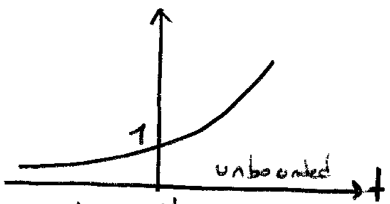
Ex. $f(t)=t$, find $t\dot{\delta}(t)$

$$t\dot{\delta}(t) = t\Big|_{t=0}\dot{\delta}(t) - \frac{d}{dt}t\Big|_{t=0}\delta(t)$$

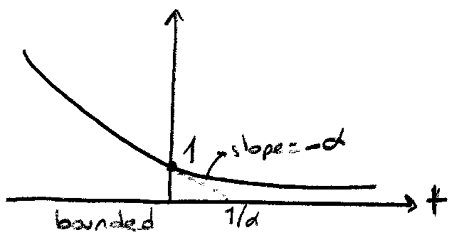
$$= 0\,\dot{\delta}(t) - 1\,\delta(t) = -\delta(t) \qquad \boxed{t\,\dot{\delta}(t) = -\delta(t)}$$

Exponential Function

$f(t) = e^{\alpha t} \quad \alpha>0$ $\qquad\qquad$ $f(t) = e^{-\alpha t} \quad \alpha>0$



$$\lim_{t\to\infty} e^{\alpha t} = \infty$$

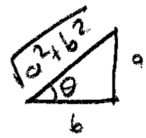


$\tau = \frac{1}{\alpha} \rightarrow$ time constant

Sinusoidal Function

$v(t) = a\sin(\omega t) + b\cos(\omega t)$

$v(t) = \sqrt{a^2+b^2}\cos\left(\omega t + \tan^{-1}\frac{a}{b}\right)$



$\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$

$\cos\theta = \frac{b}{\sqrt{a^2+b^2}}$

$\theta = \text{Arctan}\,\frac{a}{b}$

$v(t) = a\sin(\omega t) + b\cos(\omega t)$

$$= \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin(\omega t) + \frac{b}{\sqrt{a^2+b^2}}\cos(\omega t)\right]$$

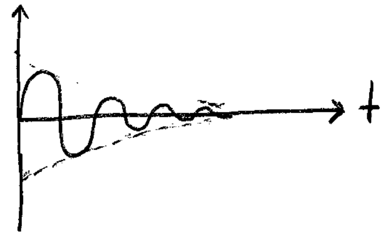
$$= \sqrt{a^2+b^2}\left[\sin\theta\sin(\omega t) + \cos\theta\cos(\omega t)\right] = \sqrt{a^2+b^2}\cos(\omega t - \theta)$$

$$= \sqrt{a^2+b^2}\cos\left(\omega t - \tan^{-1}\frac{a}{b}\right)$$

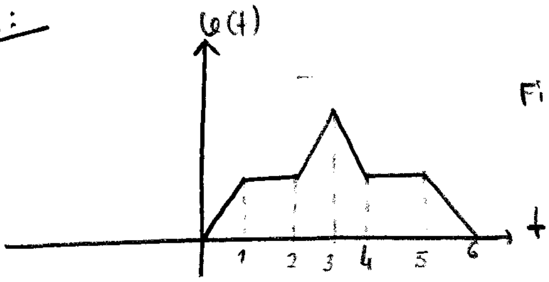
Composite Waveforms

$$f(t) = e^{-\frac{t}{\tau}} \sin(\omega t)\, u(t)$$

$$= \underset{\substack{\uparrow \\ \text{time constant}}}{e^{-\frac{t}{\tau}}} \sin\left(\frac{2\pi}{\underset{\substack{\uparrow \\ \text{period}}}{T}} t\right) u(t)$$



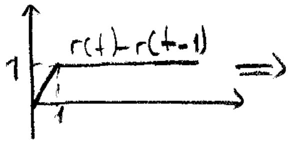
Ex:



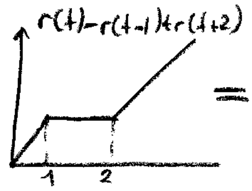
Find $v(t)$ in terms of $r(t), u(t), \delta(t) \ldots$

Soln:

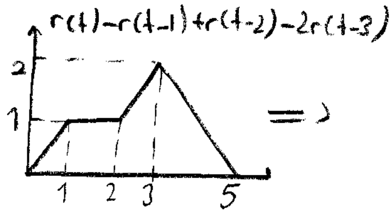
$$v(t) = r(t) - r(t-1) + r(t-2) - 2r(t-3) + r(t-4) - r(t-3) + r(t-6)$$



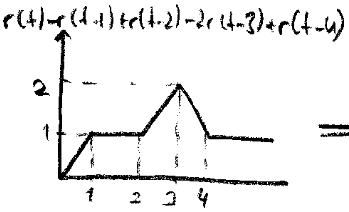
$\Rightarrow$



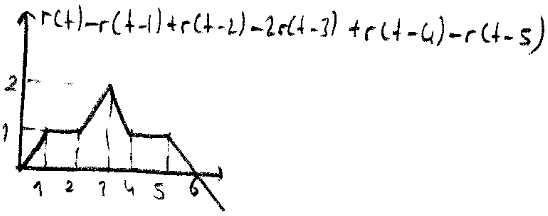
$\Rightarrow$



$\Rightarrow$

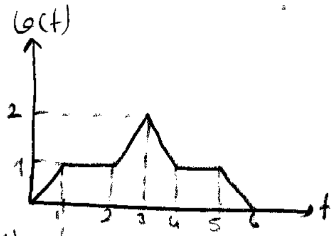


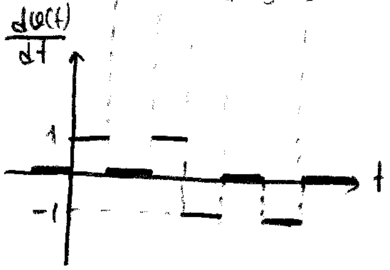
$\Rightarrow$



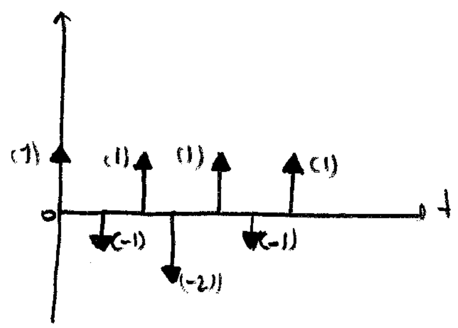
$$\frac{d\,w(t)}{dt} = u(t) - u(t-1) + u(t-2) - 2u(t-3) + u(t-4) - u(t-5) + u(t-6)$$

$$= \left[u(t) - u(t-1)\right] + \left[u(t-2) - u(t-3)\right] - \left[u(t-3) - u(t-4)\right] - \left[u(t-5) - u(t-6)\right]$$

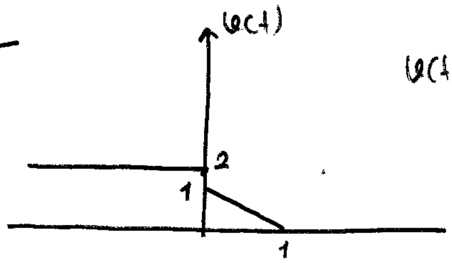




$$\frac{d^2 w(t)}{dt^2} = \delta(t) - \delta(t-1) + \delta(t-2) - 2\delta(t-3) + \delta(t-4) - \delta(t-5) + \delta(t-6)$$



Ex.

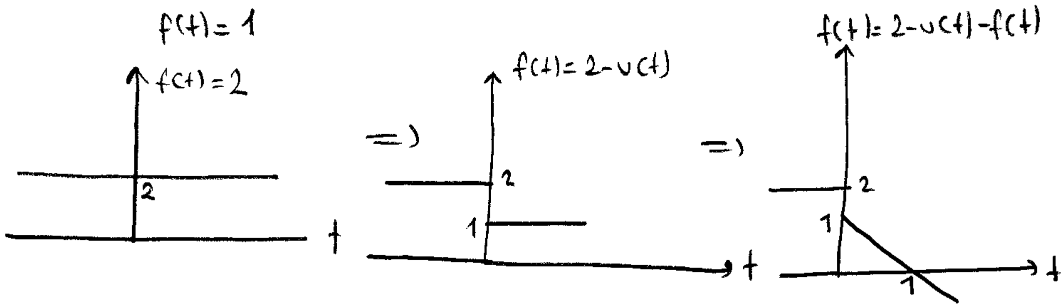


u(t)=?

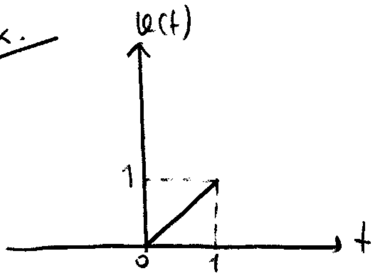
Soln.

f(t)=u(t)

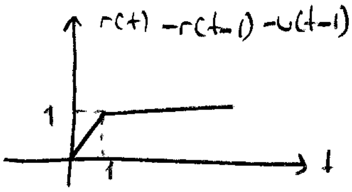
f(t)= 1



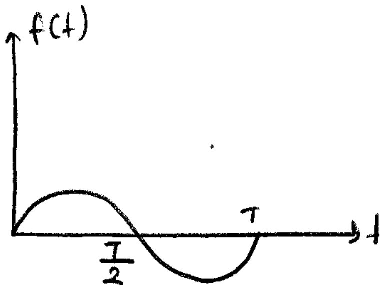
Ex.



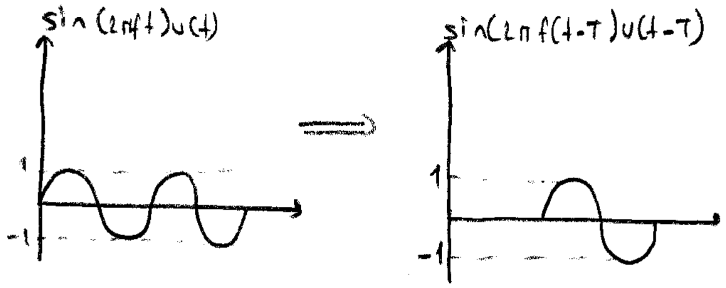
Soln.



Ex.



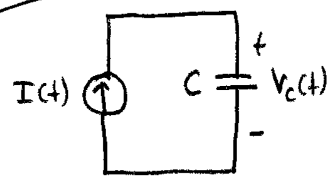
Soln:



~ Continuity Property ~

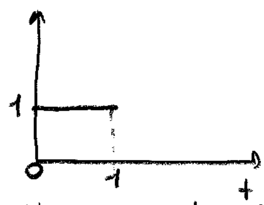
* Voltage across the capacitor is continues if the current through the capacitor is bounded.
* Current through the inductor is continoues if the voltage across the inductor is bounded.

Ex.



$V_c(0) = V_c(0) = 0$

a) $I(t) = u(t) - u(t-1)$



| when $t<1$ $I(t)=1$
$t>1$ $I(t)=0$ |

$$V_c(t) = V_c(0) + \frac{1}{c}\int_0^t I(t')dt'$$

$$= 0 + \frac{1}{c}\int_0^t 1dt = 0 + \frac{1}{c}t \quad 0<t<1$$

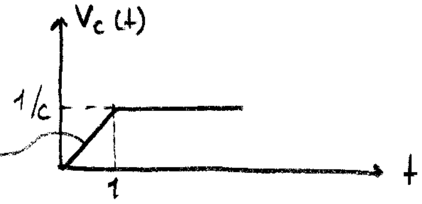
when $t>1$

$$V_c(t) = V_c(0) + \int_0^t I(t')\,dt'$$

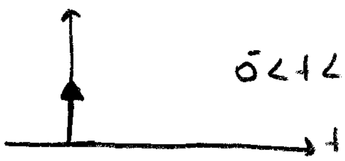
$$= V_c(0) + \frac{1}{c} \cdot \int_0^1 1\,dt + \underbrace{\int_1^t 0\,dt}_{0}$$

$$= 0 + \frac{1}{c}$$

$\frac{1}{c}t$



b) $I(t) = \delta(t)$ → unbounded input when $t=0$



$0^- < t < 0^+$

$$V_c(t) = V_c(\delta) + \frac{1}{c} \int_{0^-}^{t} \delta(t)\,dt$$

$$= 0 + \frac{1}{c} u(t) \qquad V_c(0^+) = 0 + \frac{1}{c} = \frac{1}{c}$$

$t>0$

$$V_c(t) = V_c(0^-) + \frac{1}{c} \int_{0^-}^{t} I(t')\,dt'$$

$$= 0 + \frac{1}{c} \left[\int_{0^-}^{0^+} \delta(t')\,dt' + \int_{0^-}^{t} 0\,dt \right]$$

$$= \frac{1}{c}$$