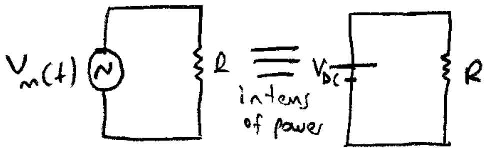
03.07.2010

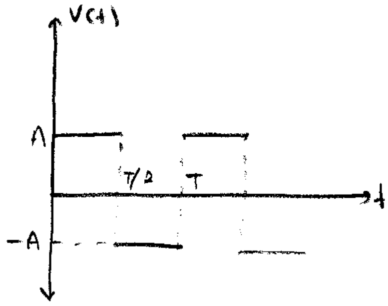
Summary

AC circuit



$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t')\,dt'}$$

Square Wave



$$V(t) = \begin{cases} A & 0 \le t < T/2 \\ -A & T/2 \le t < T \end{cases}$$

V(T) is periodic with T

Let's find average and rms values for square wave;

$$V_{avg\,sqr} = \frac{1}{T}\int_0^T v(t)\,dt$$

$$= \frac{1}{T}\left[\int_0^T A\,dt + \int_{T/2}^T -A\,dt\right]$$

$$= \frac{1}{T}\left[A\,t\Big|_0^{T/2} + (-A)\,t\Big|_{T/2}^{T}\right]$$

$$= \frac{1}{T}\left[A(T/2 - 0) + (-A)(T - T/2)\right] = 0$$

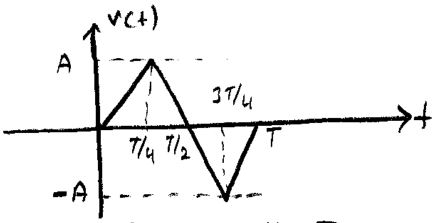
$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

$$= \sqrt{\frac{1}{T}\left[\int_0^T A^2\,dt + \int_{T/2}^T (-A)^2\,dt\right.}$$

$$= \sqrt{\frac{1}{T}\left[\int_0^T A^2\,dt + \int_{T/2}^T A^2\,dt\right.}$$

$$= \sqrt{\frac{1}{T} A^2 \left[t\Big|_0^{T/2} + t\Big|_{T/2}^{T}\right)} = A$$

Triangular Wave



Periodic with T

$$V(t) = \begin{cases} \frac{4A}{T}t & 0 \le t < T/4 \\ -\frac{4A}{T}\left[t - T/2\right] & T/4 \le t < \frac{3T}{4} \\ \frac{4A}{T}(t-T) & \frac{3T}{4} \le t < T \end{cases}$$

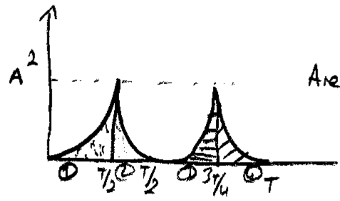
For triangular wave;

$$V_{avg_{v(t)}} = 0$$

$$V_{RMS_{v(t)}} = \frac{A}{\sqrt{3}}$$

$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T v^2(A)\,dt}$$

$V^2(t)$ second order parabols



Area ① ② ③ ④ are the same

So $$\sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt} = \sqrt{\frac{1}{T}\,4\int_0^{T/4} v^2(t)\,dt}$$

for triangular wave

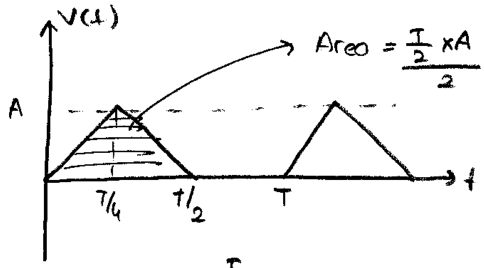
$$= \sqrt{\frac{1}{T}\,4\int_0^{T/4}\left(\frac{4A}{T}t\right)^2 dt}$$

$$= \sqrt{\frac{1}{T}\,4\,\frac{16A^2}{T^2}\int_0^{T/4} t^2\,dt}$$

$$= \sqrt{\frac{64}{T^3}\,\frac{t^3A^2}{3}\Big|_0^{T/4}}$$

$$= \sqrt{\frac{64A^2}{T^3 \times 3}\left[\frac{T^3}{64} - 0\right]} = \frac{A}{\sqrt{3}}$$

Ex.



$$\text{Area} = \frac{\frac{T}{2} \times A}{2} = \frac{AT}{4}$$

$$V_{avg_{v(t)}} = \underbrace{\frac{1}{T}\int_0^T v(t)\,dt}_{\substack{\text{find the area}\\ \text{under this singular}\\ \text{up to x-axis}}} = \frac{1}{T}\,\frac{AT}{4} = \frac{A}{4}$$

$$v(t) = \begin{cases} \frac{4A}{T}\,t & 0 \le t < T/4 \\ -\frac{4A}{T}(t - T/2) & \frac{T}{4} \le t < T/2 \\ 0 & T/2 \le t < T \end{cases}$$

$$V_{avg\,v(t)} = \frac{1}{T}\int_0^T v(t)\,dt$$

$$= \frac{1}{T}\left[\int_0^{T/4} \frac{4A}{T}\,t\,dt + \int_{T/4}^{T/2}\left(-\frac{4A}{T}\right)\left(t - \frac{T}{2}\right)dt + \int_{T/2}^{T} 0\,dt\right]$$

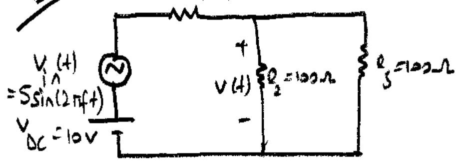
$$= \frac{1}{T}\left[\frac{4A}{T}\,\frac{t^2}{2}\Big|_0^{T/4} + \left(-\frac{4A}{T}\right)\left(\frac{t^2}{2} - \frac{T}{2}t\right)\Big|_{T/4}^{T/2} + 0\right]$$

$$= \frac{1}{T}\left[\frac{2A}{T}\left[\frac{T^2}{16} - 0\right] + \left(-\frac{4A}{T}\right)\left[\frac{1}{2}\left(\frac{T^2}{4} - \frac{T^2}{16}\right) - \frac{T}{2}\left(\frac{T}{2} - \frac{T}{4}\right)\right]\right]$$

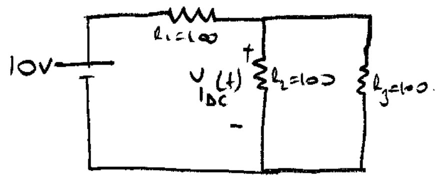
$$= \frac{1}{T}\left[\frac{2AT}{16} + \left(-\frac{4A}{T}\right)\left[\frac{1}{2}\,\frac{3T^2}{16} - \frac{T^2}{8}\right]\right]$$

$$= \frac{1}{T}\left[\frac{AT}{8} + \left(-\frac{4A}{T}\right)\left[\frac{3T^2}{32} - \frac{4T^2}{32}\right]\right] = \frac{A}{8} + \frac{4A}{32} = \frac{A}{4}$$

Ex:

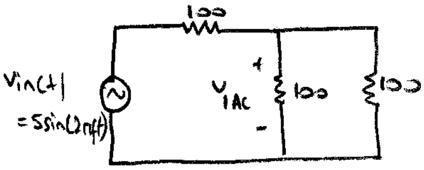


Kill $V_{in}(t)$ (only DC circuit)



$V_{1DC}(T) = \frac{10}{3}$

Kill V_{DC} (only AC circuit)



$V_{AC}(t) = \frac{5}{3}\sin(2\pi f t)$

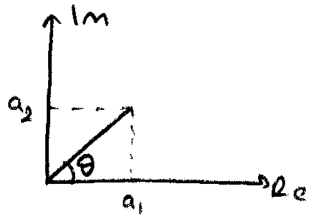
$V(t) = V_{DC} + V_{AC} = \frac{10}{3} + \frac{5}{3}\sin(2\pi f t)$

Complex Domain Operations

$A = a_1 + a_2 j$ $Re\{A\} = a_1$

$Im\{A\} = a_2$

Cartesian Representation



$A = \underbrace{\sqrt{a_1^2 + a_2^2}}_{\text{magnitude}}\, e^{j\theta}$

$e^{j\theta} = \cos\theta + j\sin\theta$

$\cos\theta = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}$

$\underbrace{A = \sqrt{a_1^2 + a_2^2}\, e^{j\theta}}_{\text{polar representation}}$

$\sin\theta = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}$

$\theta = \angle A$ = phase

$A = a_1 + a_2 j \qquad B = b_1 + b_2 j$

$A + B = (a_1 + b_1) + j\,(a_2 + b_2)$

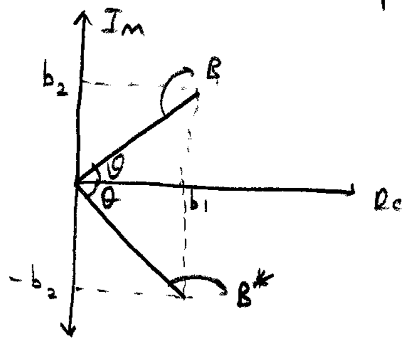
$A - B = (a_1 - b_1) + j\,(a_2 - b_2)$

$$\frac{A}{B} = \frac{a_1 + a_2 j}{b_1 + b_2 j} = \frac{(a_1 + a_2 j)(b_1 - b_2 j)}{b_1^2 + b_2^2}$$

$B = b_1 + b_2 j = \sqrt{b_1^2 + b_2^2}\, e^{j\theta} \qquad \angle B = \theta$

$B = b_1 - b_2 j \rightarrow$ conjugate $= \sqrt{b_1^2 + b_2^2}\, e^{-j\theta}$

$\angle B^* = -\theta$



$A = |A|\, e^{j\theta} \searrow$ polar

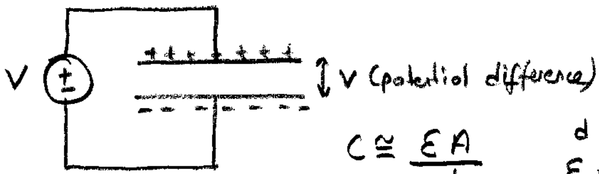
$B = |B|\, e^{j\beta} \nearrow$

$A \times B = |AB|\, e^{j(\theta + \beta)}$

$$\frac{A}{B} = \left|\frac{A}{B}\right| e^{j(\theta - \beta)}$$

Capacitors

Two terminal element used to store electrical energy.

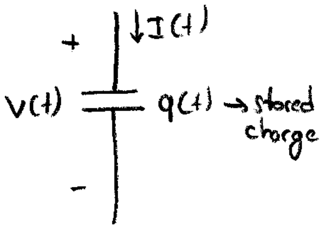


$$C \cong \frac{\varepsilon A}{d}$$

d → distance btw. plates

ε → dielectric constant

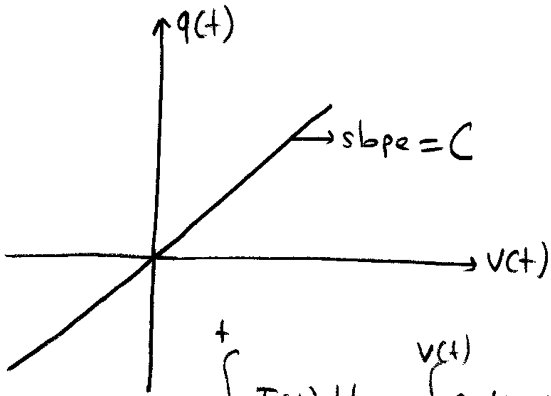
A → active area of the plates



$$I(t) = \frac{dq(t)}{dt} \rightarrow \int_{t_0}^{t} I(t)\,dt = \int_{q_1}^{q(t)} dq(t)$$

$$\int_{t_0}^{t} I(t)\,dt = q(t) - q_0$$

For LTI Capacitors



$$q(t) = C\,v(t) \quad (\text{for LTI capacitors})$$

$$\frac{dq(t)}{dt} = C\,\frac{dv(t)}{dt}$$

$$I(t) = C\,\frac{dv(t)}{dt}$$

$$\int_{t_0}^{t} I(t)\,dt = \int_{v_0}^{v(t)} C\,dv(t)$$

$$\int_{t_0}^{t} I(t')\,dt' = C\left[v(t) - v_0\right]$$

$$\boxed{v(t) = v_0 + \frac{1}{C}\int_{t_0}^{t} I(t')\,dt'}$$

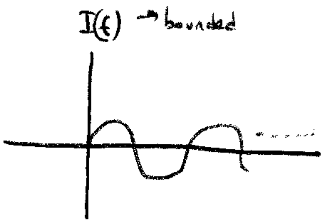
- $v(t)$ depends on $I(t)$, v_0, C
- A capacitor has a memory.

* If $v(t) = K$ (constant DC value) then

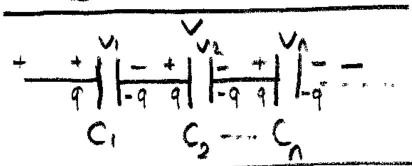
$$I(t) = C\,\frac{dK}{dt} = 0 \;//$$

(Capacitor acts as open circuit for DC voltage excitation)

* If $I(t)$ is bounded, i.e. $\|I(t)\| \le M$ for $\forall t$ then, Voltage across the capacitor is a continuous function.

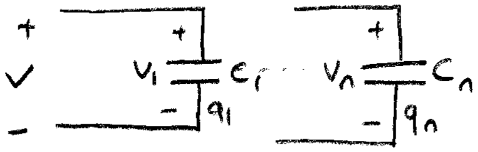


Series Connection of Capacitors



$$\boxed{\frac{1}{C_{Total}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots \frac{1}{C_n}}$$

Parallel Connection of Capacitors



$V = V_1 = V_2 = \cdots\cdots = V_n$

$q_{Total} = q_1 + \cdots + q_n$

$$\boxed{C_{total} = C_1 + \cdots + C_n}$$

~ Power and Energy ~

$$P(t) = V(t)I(t) = V(t)\, C \overbrace{\frac{dV(t)}{dt}}^{I(t)}$$

instantenious power — for LTI capacitors

$$= \frac{1}{2}\frac{d}{dt}\left[CV^2(t)\right] = P(t)$$

$$W(t,t_0) = \int_{t_0}^{t} P(t')\,dt'$$

The work done depends on initial time and final time.

$$= \int_{t_0}^{t} \frac{1}{2} C \frac{d}{dt}\left[V^2(t)\right] dt'$$

$$= \int_{V_0}^{V(t)} \frac{1}{2} C\, dV^2(t)$$

$$= \frac{1}{2} C\left[V^2(t) - V_0^2\right]$$

$$W(t,t_0) = \frac{1}{2}CV^2(t) - \frac{1}{2}CV_0^2$$

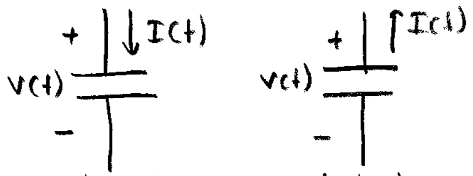
If $V_0^2 > V^2(t) \implies w(t,t_0) < 0 \implies$ Capacitor is being decharged.

$w(t,t_0) > 0 \implies$ " " " charged

* If initially capacitor is uncharged (Let $t_0=0$ $q_0=0$ and $V_0=0$)

$$E(t) = \frac{1}{2} C V^2(t)$$

* If $P(t) > 0 \implies$ Capacitor is being charged (string energy)
* If $P(t) < 0 \implies$ " " " decharged (supplying energ to its circuit)



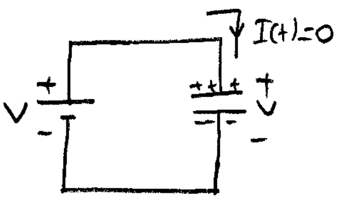
charged $P(t)>0$ | decharging $P(t)<0$

* Capacitor is a lossless energy storage element.

$$E(t) = \frac{1}{2} C V^2(t) \geq 0 \text{ when } C > 0$$

Capacitor is a passive element.

<u>Capacitors in DC Circuits</u>



* A capacitor blocks DC Current
* It acts as open circuit when DC voltage is applied.

Capacitors in AC Circuits

$$Q = CV \qquad I = C\frac{dV}{dt}$$

In Laplace Domain

$$I(s) = Cs\,V(s)$$

$$\frac{V(s)}{I(s)} = \frac{1}{Cs} \xrightarrow[s=j\omega]{\text{in frequency domain}} \frac{V(j\omega)}{I(j\omega)} = \frac{1}{j\omega C} = \frac{1}{j2\pi f C}$$

$$\underbrace{\omega}_{\text{angular frequency}}^{\frac{rad}{sec}} = 2\pi \underbrace{f}_{\text{frequency}}^{\frac{1}{sec} = \text{Hertz}}$$

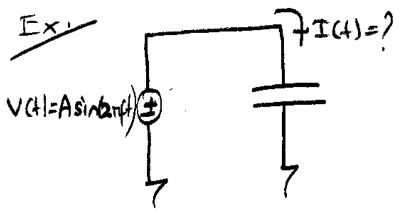
$$\frac{V(j\omega)}{I(j\omega)} = \frac{-j}{\omega C} = -jX_C$$

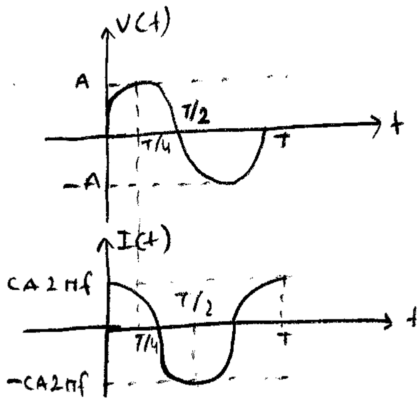
$X_C \rightarrow$ Complex resistance

$$X_C = \frac{1}{2\pi f C}$$ $\Rightarrow$ When f increase $|X_C|$ decrease and much more current will pass over the capacitor.

* A capacitor acts as short circuit for AC excitation.

Ex:





$$I(t) = C\frac{dV(t)}{dt}$$

$$= C\frac{d}{dt}A\sin(2\pi f t)$$

$$= CA\,2\pi f\cos(2\pi f t)$$

$$I(t) = CA\,2\pi f\sin(2\pi f t + 90^\circ)$$

* Current leads Voltage 90° for capacitor.

Inductors

Any device that stores magnetical energy is called as inductor. For inductors the flux over the inductor, is a function of the current passing through the inductor.

$$\underset{\text{webers}}{\phi(t)} = \phi(I(t), t)$$

$$\frac{d\phi}{dt} = V(t) \quad \text{(Faraday's Law)}$$

LTI Inductors

$$\phi = L I$$

Constant ↳ Inductance Value (Henry)

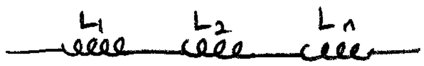
$$\frac{d\phi}{dt} = L\frac{dI}{dt} \implies \boxed{V = L\frac{dI}{dt}} \longrightarrow I(t) = I_0 + \frac{1}{L}\int_{t_0}^{t} V(t')\,dt'$$

* The current over the inductor is a continuous function of time if the voltage across the inductor is bounded for $\forall t$.

* Assume $I(t) = K$ (constant DC excitation) $V = L\frac{dI}{dt} = L\frac{dK}{dt} = 0$

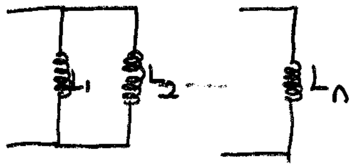
↳ An inductor behaves as short circuit for DC current excitation.

Series Connection of Inductors



$$\boxed{L_{Total} = L_1 + L_2 + \cdots L_n}$$

Parallel Connection of Inductors



$$\boxed{\frac{1}{L_{Total}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}}$$

~ Power and Energy ~

$$P(t) = V(t)I(t) \overset{\text{LTI Inductor}}{=} L\frac{dI(t)}{dt}I(t)$$

instatentious power

$$P(t) = \frac{1}{2}L\left[\frac{d}{dt}I^2(t)\right]$$

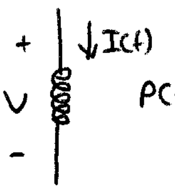
$$W(t,t_0) = \int_0^t P(t')dt' = \int_{t_0}^{t}\frac{1}{2}L\left[\frac{d}{dt}L^2(t)'\right]dt'$$

$$= \int_{I_0}^{I_t}\frac{1}{2}L\,dI^2(t) = \frac{1}{2}L\left[I^2(t) - I_0^2\right]$$

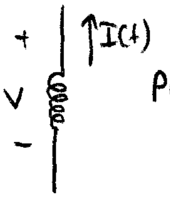
$$= \frac{1}{2}LI^2(t) - \frac{1}{2}LI_0^2$$

if $I^2(t) > I_0^2$ inductor is being charged

if $I^2(t) < I_0^2$ " " " decharging.



$P(t) > 0$ (being charged)



$P(t) < 0$ (decharging)