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AC Circuits:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

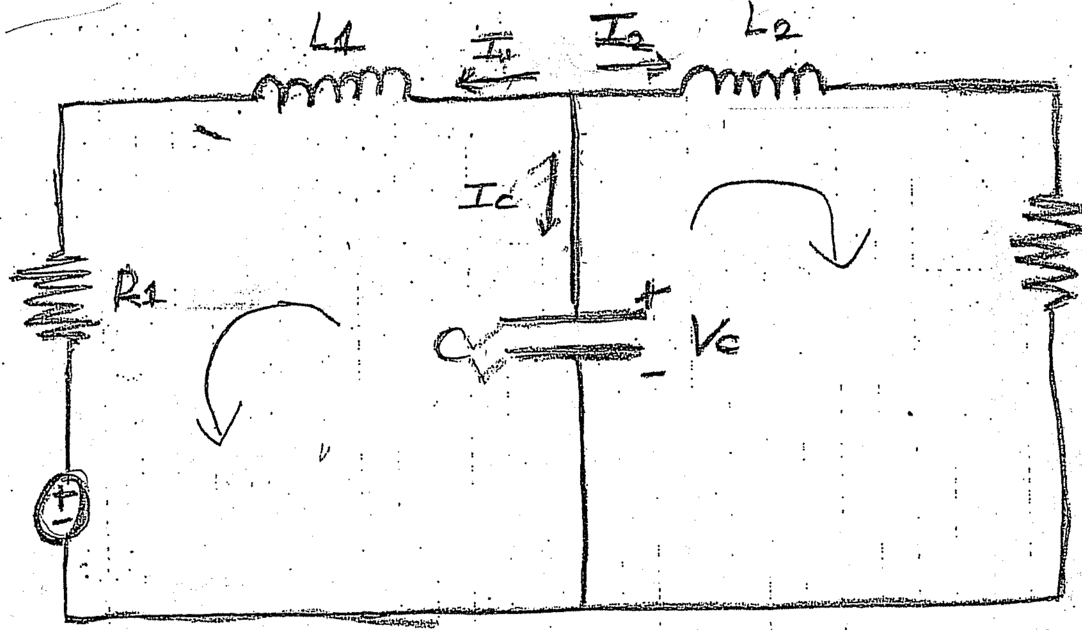
state space representation

derivatives of state variables

state variables

input vector (source vector)

Ex:



V_c, I_1, I_2 $\longleftrightarrow$ states

$$-I_1 - I_2 = I_C$$

$$I_1 + I_2 + C\frac{dV_c}{dt} = 0$$

$$I_1 + I_2 + I_C = 0$$

$$I_1 + I_2 + C\,\frac{dV_c}{dt} = 0$$

$$V_{L_1} + V_{R_1} + V_S = V_C$$

$$V_{L_2} + V_{R_3} = V_C$$

$$\Longrightarrow \boxed{\frac{dV_c}{dt} = -\frac{I_1}{C} - \frac{I_2}{C}}$$

$$L_1\frac{dI_1}{dt} + V_{R_1} + V_S = V_C$$

$$L_2\frac{dI_2}{dt} + V_{R_3} = V_C$$

$$\frac{dI_2}{dt} = \frac{V_C}{L_2} - \frac{V_{R3}}{L_2}$$

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$$V_{L_1} + V_{R_1} + V_S = V_C$$

$$L_1 \frac{dI_1}{dt} + R_1 I_1 + V_S = V_C$$

$$\Longrightarrow \boxed{\frac{dI_1}{dt} = \frac{-V_S}{L_1} - \frac{R_1}{L_1} I_1 + \frac{V_C}{L_1}}$$

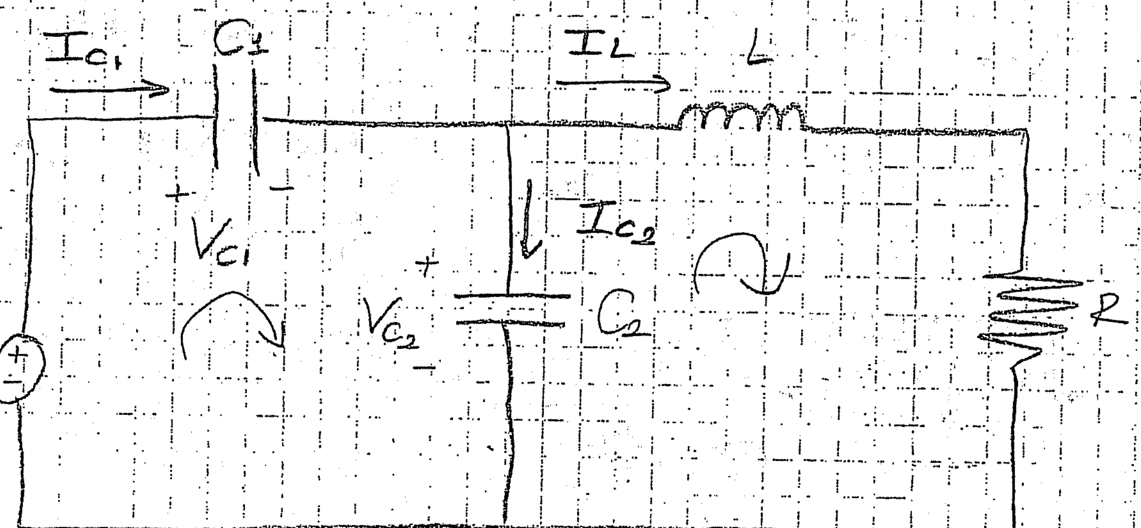
$$V_C = V_{L_2} + V_{R_3}$$

$$V_C = L_2 \frac{dI_2}{dt} + I_2 . R_3$$

$$\Longrightarrow \boxed{\frac{dI_2}{dt} = \frac{V_C}{L_2} - \frac{R_3}{L_2} I_2}$$

$$? \begin{bmatrix} \dot{V}_C \\ \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/C & -1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ 1/L_2 & 0 & -R_3/L_2 \end{bmatrix} \begin{bmatrix} V_C \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} V_S$$

Ex:



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$$I_{C1} = I_{C2} + I_L$$

$$C_1 \frac{dV_{C1}}{dt} = C_2 \frac{dV_{C2}}{dt} + I_L$$

$$\boxed{V_S = V_{C1} + V_{C2}}$$

$$I_{C1} = C_1 \frac{dV_{C1}}{dt} \Longrightarrow I_{C1} = C_1 \frac{d}{dt}\left[V_S - V_{C2}\right]$$

$$C_2 \frac{dV_{C2}}{dt} + I_L = C_1 \frac{d}{dt}\left[V_S - V_{C2}\right]$$

$$(C_1 + C_2) \frac{dV_{C2}}{dt} = C_1 \frac{dV_S}{dt} - I_L$$

$$\boxed{\frac{dV_{C2}}{dt} = \frac{C_1}{C_1 + C_2} \frac{dV_S}{dt} - \frac{1}{C_1 + C_2} I_L}$$

$$V_{C2} = V_L + V_R$$

$$V_{C2} = L \frac{dI_L}{dt} + R . I_L \Longrightarrow$$

$$\boxed{\frac{dI_L}{dt} = \frac{V_{C2}}{L} - \frac{R}{L} I_L}$$

?

$$\begin{bmatrix} \dot{V}_{C2} \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} 0 & -1/(C_1+C_2) \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} V_{C2} \\ I_L \end{bmatrix} + \begin{bmatrix} C_1/(C_1+C_2) \\ 0 \end{bmatrix} \frac{dV_S}{dt}$$

$$\boxed{I_{C1} - I_{C2} = I_L}$$

$$-V_S + V_{C1} + V_{C2} = 0$$

$$V_L + V_R = V_{C2}$$

* Either V_{C1} or V_{C2} can be written in terms of the other variable without any state equation using $\boxed{-V_S + V_{C1} + V_{C2} = 0}$

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Analysis of N^{th} order circuits:

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_n x = b_0 \frac{d^m u(t)}{dt^m} + b_1 \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_m u(t)$$

requires N initial conditions

$$x(0);\ \frac{dx}{dt}(0);\ \cdots\ ;\ \frac{d^{n-1}x(0)}{dt^{n-1}}$$

$$x(t) = X_{\text{zero-state}}(t) + X_{\text{zero-input}}(t)$$

zero-state: due to input only ($u(t)$) and initial conditions are zero $\left(x(0) = 0;\ \frac{dx}{dt}(0) = 0; \ldots ; \frac{d^{n-1}x}{dt^{n-1}}(0) = 0 \right)$

zero-input: due to initial conditions (initial conditions are non-zero) but $u(t) = 0$

$x(t) \longrightarrow$ state

$u(t) \longrightarrow$ input

Let's find zero-input solution

assume $u(t) = 0$, instead of "$\frac{d}{dt}$" terms put "s"

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

$\longrightarrow$ characteristic equation

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* Zeros (roots) of the characteristic equation are the natural frequencies of the circuit.

* If all the natural frequencies of the circuit are distinct (different from each other)

$$X_{homogeneous} = X_{zero-input} = \sum_{i=1}^{n} K_i \, e^{s_i t}$$

$n \rightarrow$ order of chac. eqn.

(not always true)

$\rightarrow$ parameters unknown due to initial condition

$\rightarrow$ parameters are evaluated

$$s_i \neq s_k \quad \text{for all} \quad (1 < i < n,\ 1 < k < n)$$

s_i = natural frequency

K_i = constant to be determined by initial conditions

* If there are some natural frequencies with are equal to each other. (for some i and k $s_i = s_k$)

$\rightarrow$ meaning some natural frequency has a multiplicity.

(Ex: $s_1 = -2$, $s_2 = -2$, $s_4 = -2$

$s_1 = s_2 = s_4$ multiplicity of $s = -2$ is 3)

$$X_h(t) = \sum_{i=1}^{N} \sum_{m=0}^{d_k - 1} K_{im} \, t^m \, e^{s_k t}$$

$N = \#$ of distinct roots

$\#$ → number

s_k = natural frequencies with multiplicities d_k

Ex: $(s-1)(s-1)(s-2)(s+j)(s-j)(s+j)(s-j) = 0$

$s=1_n$ $s=2$ $-j$ j $-j$ j

7 roots

4 distinct roots

multiplicity of $s_1 = 1$ 2

" " $s_2 = 2$ 1

" " $s_3 = -j$ 2

" " $s_4 = j$ 2

$$= K_{10}\, t e^{t} + K_{11}\, e^{t} + K_{20}\, e^{2t} + K_{30}\, t e^{-jt} + K_{31}\, e^{-jt}$$

$$+ K_{40}\, t e^{jt} + K_{41}\, e^{jt}$$

Complete Response:

$X(t) = X_h(t) + X_p(t)$ (generally $X_p(t)$ (particular solution) is similar to input $u(t)$)

Let $\underbrace{u(t)}_{\text{input}} = \underbrace{V_0}_{\text{amplitude}} \cos(\omega t + \theta)$

θ → phase

ω → angular frequency

what is $X_p(t)$?

2 situation

① $\boxed{s_k \neq j\omega}$ for $k = 1, ----, N$

s_k: any natural frequency

$j\omega$: angular frequency of input

$$X_p(t) = A_m \cos(\omega t + \theta + \phi)$$

A_m: amplitude will change

ω: angular frequency is same

ϕ: extra phase difference

② $s_k = j\omega$ for some k (no real part for s_k exist)

s_k is purely imaginary

$s_k = 0 + jb$

$$X_p(t) = t^{d_k} A_m \cos(\omega t + \theta + \phi)$$

d_k is the multiplicity of s_k.

Characteristic of complete solution (giving same input $u(t) = V_0 \cos(\omega t + \theta)$)

① K_{im} values are determined according to initial conditions such that $X_h(t) = 0$ in that case

$$X(t) = \underbrace{X_h(t)}_{0} + X_p(t)$$

$X(t) = X_p(t)$ (since $X_h(t) = 0$ there will be no trasient solution.)

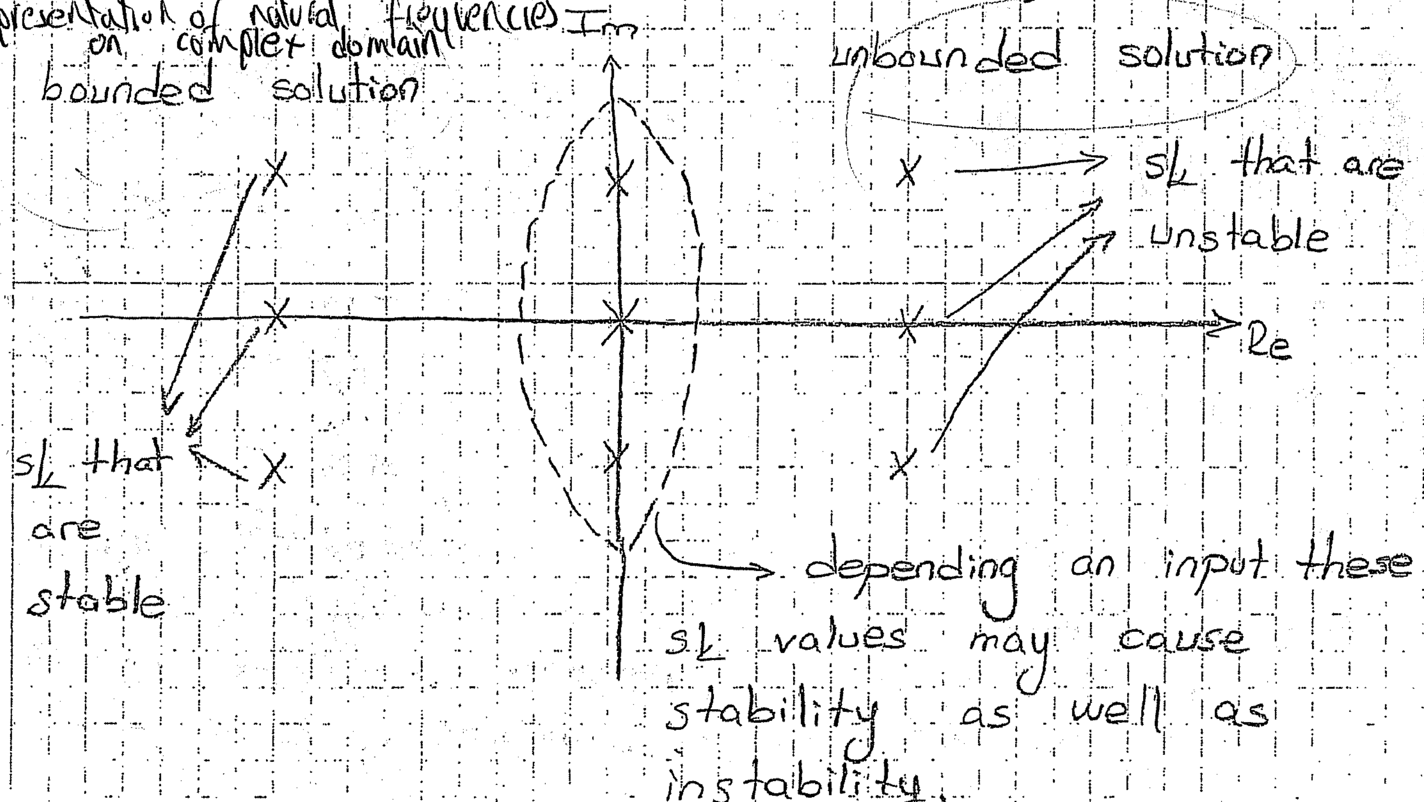
② $Re\{s_k\} < 0$ $k = 1, ---, N$

as $t \longrightarrow \infty$, $|s^{kt}| \longrightarrow 0$; homogeneous solution will go to 0 as $t \longrightarrow \infty$ and only particular solution will remain. (Steady-state) (SS)
$\hookrightarrow X_p(t)$

Special case $(u(t) = V_0 \cos(\omega t + \theta))$

as $t \longrightarrow \infty$ $x(t) = X_p(t) \longrightarrow$ sinusoidal steady state solution is obtained (SSS)

Representation of natural frequencies on complex domain



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③ $Re\{s_k\} \leq 0 \quad k=1,\dots,N$ } same roots (zeros) have zero real parts } and $u(t) = V_0 \cos(\omega t + \theta)$

same input

a) $s_k \neq j\omega$

let $X_{h_0}(t)$ be a linear combination of sinusoidals due to roots with zero real part

$$X(t) = X_h(t) + X_p(t)$$

$$X_h(t) = \underbrace{X_{h_0}(t)}_{\substack{\text{some sinusoidals} \\ + \text{ some DC terms}}} + \underbrace{X_{h_{non\text{-}0}}(t)}_{\text{decaying terms}}$$

$$X(t) = X_h(t) + X_p(t)$$

$\hookrightarrow A_m \cos(\omega t + \theta + \phi)$

as $t \to \infty$; $\boxed{X(t) = X_{h_0}(t) + \underbrace{A_m \cos(\omega t + \theta + \phi)}_{X_p(t)}}$

$X_p(t)$ → SSS (sinusoidal steady state)

SS → steady state part

b) $s_k = j\omega \longrightarrow$ unbounded situation

$$X_p(t) = t^{d_k} A_m \cos(\omega t + \theta + \emptyset)$$

④ $Re\{s_i\} > 0$ for some s_i (not important to look at the input)

$x(t)$ is unstable

⑤ $Re\{s_i\} = 0$ and $s_i = j\omega_0$ but s_i has a multiplicity, $X(t) \longrightarrow$ unbounded

$j\omega_0$: different from input angular frequency ω ($\omega_0 \neq \omega$)

unbounded solution $\longrightarrow$ no steady state

Case 2:

$$D = \frac{d}{dt}$$

$$D^1 = \frac{d}{dt}$$

$$D^2 = \frac{d^2}{dt^2}$$

$$(D^3 + 4D^2 + 6D + 4)\, X(t) = u(t)$$

char - equation $= s^3 + 4s^2 + 6s + 4 = 0$

$$(s+1+j)(s+1-j)(s+2) = 0$$

$s_1 = -1-j$

$s_2 = -1+j$ → all multiplicities are 1

$s_3 = -2$

$Re\{s_1\} < 0 \quad Re\{s_2\} < 0 \quad Re\{s_3\} < 0$

2^{nd} cased

$$X_h(t) = \underbrace{A_1 e^{-2t}}_{\text{due to } s_3=-2} + \underbrace{A_2 e^{-t}\cos(t) + A_3 e^{-t}\sin(t)}_{\text{due to } s_1 = s_2^*}$$

$s_1 = -1+j \quad s_2 = -1-j$

assume $u(t) = 4\cos(2t+30°)$

$$X_p(t) = K\cos(2t+\emptyset) = K_1\cos(2t) + K_2\sin(2t)$$

put this function in diff - eqn. and find K_1 and K_2

$$X_p(t) = \frac{1}{\sqrt{10}}\cos\Big(2t+30\underbrace{-180° + \tan^{-1}\left(\tfrac{1}{3}\right)}_{?}\Big)$$

$$X(t) = X_h(t) + X_p(t)$$

$$\lim_{t\to\infty} X(t) = \underbrace{X_p(t)}_{\text{bounded}} \quad SSS$$

Case 3a

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Ex: $(D^3+D^2+D+1)\,X(t)=u(t)=4\cos(2t+30^\circ)$

$\omega=2$

$s\angle \mp j2$

$s^3+s^2+s+1=0=(s+1)(s+j)(s-j)=0$

$s_1=-1$ $\quad Re\{s_1\}<0$

$s_2=-j$ $\quad Re\{s_2\}=0$ $\quad$ Case 3a is provided

$s_3=j$ $\quad Re\{s_3\}=0$

$X_h(t)=K_1e^{-t}+K_2\cos t+K_3\sin t$ $\quad \mp j$

$X_p(t)=\dfrac{4}{\sqrt{45}}\cos\Big(2t+30^\circ+\underbrace{180^\circ-\tan^{-1}(2)}_{?}\Big)$

$X(t)=X_h(t)+X_p(t)$

as $t\to\infty$ $\quad X(t)=\underbrace{K_2\cos(t)+K_3\sin(t)+\underbrace{X_p(t)}_{SSS}}_{SS}$

Case 3b:

$(D^3+D^2+4D+4)\,X(t)=u(t)=4\cos(2t+30^\circ)$

$s^3+s^2+4s+4=0$

$(s+2j)(s-2j)(s+1)=0$

$s_1=2j \quad s_2=-2j \quad s_3=-1$

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$Re\{s_1\} = Re\{s_2\} = 0 \quad Re\{s_3\} < 0$ $\quad \boxed{\omega = 2}$

$\boxed{s_1 = 2j = j\omega}$ $\longrightarrow$ case 3b

$X_h(t) = K_1 e^{-t} + K_2 \cos(2t) + K_3 \sin(2t)$

$X_p(t) = M t \cos(2t + \phi)$

$X_p(t) = t\left[M_1 \cos 2t + M_2 \sin 2t\right]$

$X_p(t) = \frac{1}{\sqrt{5}} \cdot \cos\left(2t + 30 - \underbrace{180° + \tan^{-1}\left(\frac{1}{2}\right)}_{2}\right)$

$X(t) = X_h(t) + X_p(t)$

as $t \longrightarrow \infty$ $\quad X(t) = K_2 \cos(2t) + K_3 \sin(2t) + \underbrace{X_p(t)}_{\text{unbounded}}$

$X(t) \longrightarrow \mp\infty$

no steady-state solution exist

4^{th} case:

Ex: $(s-1)(s+1)(s+j)(s-j) = 0$

$u(t) = 4\cos(2t + 30°)$

$s_1 = 1 \quad s_3 = -j \quad K_1 e^{t} + K_2 e^{-t} + K_3 \cos(t) + K_4 \sin(t)$

$s_2 = -1 \quad s_4 = j$

$$X_h(t) = K_1 e^{t} + K_2 e^{-t} + K_3 \cos t + K_4 \sin t$$

$$X_p(t) = M \left[\cos(2t + \phi) \right]$$

as $t \longrightarrow \infty$; $X(t) \longrightarrow \mp \infty$

Case 5:

$$(D^5 + D^4 + 2D^3 + 2D^2 + D + 1)\, X(t) = u(t) = 4 \cos(2t + 30^\circ)$$

$$(s+1)(s+j)^2 (s-j)^2 = 0$$

$s_1 = -1$ $\quad s_2 = -j$ $\quad s_3 = j$

multp = 2 $\quad$ multp = 2

$$X_h(t) = K_1 e^{-t} + K_2 \cos(t) + K_3 \sin(t) + \underbrace{K_4\, t \cos t}$$
$$+ \underbrace{K_5\, t \sin t}$$

due to multiplicity s_2 and s_3

$s_k \neq j\omega = 2j \qquad k = 1, 2, 3$

$$X_p(t) = M \cos(2t + \phi)$$

$$= \frac{4}{\sqrt{2197}} \cos\left(2t + 30^\circ - \tan^{-1}\left(\frac{46}{9}\right)\right)$$

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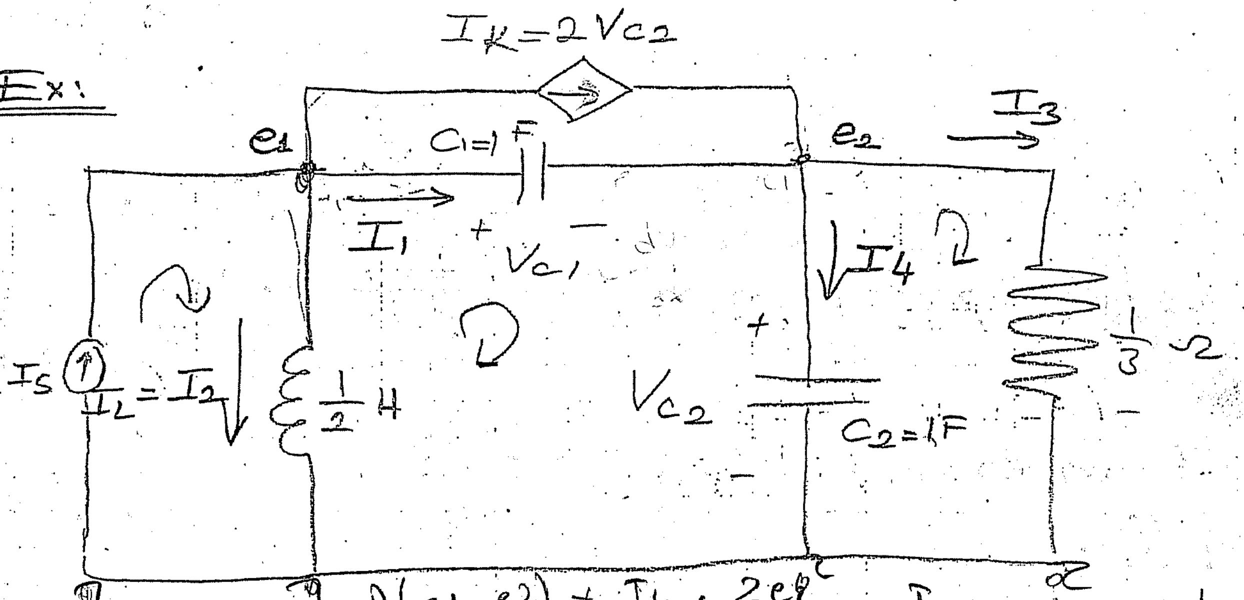
$I_K = 2V_{c2}$

Ex:



$D(e_1-e_2) + \frac{I_L}{c} + \frac{2e_2}{C_1} = \frac{I_s}{C_1}$ $D \to \frac{d}{dt}$

$\sqrt{}\ I_s = I_1 + I_2 + 2V_{c2}$

$$\boxed{I_s = D(e_1 - e_2) + I_L + 2e_2} \quad (1)$$

$\sqrt{}\ I_1 + I_K = I_4 + I_3$

$$\boxed{1.D(e_1 - e_2) + 2e_2 = 1.D.e_2 + \frac{e_2}{1/3}} \quad (2)$$

$$(3) \quad \boxed{e_1 = \frac{1}{2} D I_L}$$

(1) $De_1 - De_2 + I_L = I_s$

$e_1(D) + e_2(-D+2) + I_L(1) = I_s$

(2) $De_1 + (-2D-1)e_2 = 0$

(3) $\frac{1}{2}De_1 - I_L = 0$

$$\underbrace{\begin{array}{ccc} e_1 & e_1 & I_L \\ \left[\begin{matrix} D & -D+2 & 1 \\ D & -2D-1 & 0 \\ -1 & 0 & \frac{1}{2}D \end{matrix}\right] \end{array}}_{P(D)} \begin{bmatrix} e_1 \\ e_2 \\ I_L \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \\ 0 \end{bmatrix}$$

$V_L = L\frac{dI_L}{dt}$

$e_1 = \frac{L}{2}\frac{dI_L}{dt}$

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Let's find det (P(D))

$$D\left[(-2D-1)\frac{1}{2}D-0\right]-(-D+2)\left[\frac{1}{2}D.D-0\right]$$

$$+1\left[-(-2D-1)(-1)\right]$$

$$-D^3-\frac{D^2}{2}+\frac{D^3}{2}-D^2-2D-1$$

$$-\frac{D^3}{2}-\frac{3}{2}D^2-2D-1 = \det\left(P(D)\right)$$

* Natural frequencies of LTI network (circuit) are the roots of the polynomial $\det(P(D))=0$ where $P(D)$ is the matrix with differential equation terms that describe the network.

instead of D put s in the polynomial

$$-\frac{s^3}{2}-\frac{3}{2}s^2-2s-1=0$$

$$-(s+1)(s^2+2s+2)=0$$

$Re\{s_i\}<0$

$Re\{s_i\}<0$

$s_1=-1 \quad s_{2,3}=-1\mp j$

Assume $I_s=0$ (no input)

Natural response for all states

$$\begin{bmatrix} e_1 \\ e_2 \\ I_L \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{(-1+j)t} + c_3 e^{(-1-j)t} \\ c_4 e^{-t} + c_5 e^{(-1+j)t} + c_6 e^{(-1-j)t} \\ c_7 e^{-t} + c_8 e^{(-1+j)t} + c_9 e^{(-1-j)t} \end{bmatrix}$$

3 initial conditions are necessary

$$e_1(0) = K_1 = c_1 + c_2 + c_3$$
$$e_2(0) = K_2 = c_4 + c_5 + c_6$$
$$I_L(0) = K_3 = c_7 + c_8 + c_9$$

(3) $\frac{1}{2} D I_L = e_1$ $\qquad \frac{1}{2}\frac{d}{dt} I_L = e_1$

$$\frac{1}{2}\frac{d}{dt} I_L(0) = e_1(0)$$

$$\frac{d}{dt} I_L(0) = 2 e_1(0)$$

$$\begin{bmatrix} e_1 \\ e_2 \\ I_L \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + e^{-t}(a_1 \cos(t) + a_2 \sin(t)) \\ c_4 e^{-t} + e^{-t}(a_3 \cos(t) + a_4 \sin(t)) \\ c_7 e^{-t} + e^{-t}(a_5 \cos(t) + a_6 \sin(t)) \end{bmatrix}$$

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Solution of State equation with no input

$$\text{(1)}\quad \dot{x} = Ax + \cancel{Bu} \qquad u = 0$$

$$X(0) = x_0$$

no input

* First find the natural frequencies (s_i values)

(1) Find the roots of $\det(sI - A) = 0$

(2) Find the eigenvalues of A

$$\boxed{AX_i = s_i X_i}$$

s_i = eigenvalue

x_i = eigenvector

Solution of (1) is

$$X_{solution}(t) = \underbrace{e^{At}}_{\text{Matrix}} \underbrace{x_0}_{\text{Vector}} \quad \text{(Natural response)}$$

$$e^{At} = I + \frac{At}{1!} + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$\mathcal{L}\{x' = Ax\}$$

$$sIX(s) - Ix(0) = AX(s)$$

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$$sI\,X(s) - AX(s) = x(0)$$

$$(sI - A)\,X(s) = x(0)$$

$$X(s) = (sI - A)^{-1}x(0) \xrightarrow{\mathcal{L}^{-1}} x(t) = L^{-1}\left[(sI - A)^{-1}\right]X_0(0)$$

$$= e^{At}X_0(0)$$

$e^{At} =$

if A has distinct eigenvalues $s_1, \ldots, s_n$

$$e^{At} = \begin{bmatrix} c_{11}e^{s_1t} & c_{12}e^{s_2t} & \cdots & c_{1n}e^{s_nt} \\ c_{21}e^{s_1t} & c_{22}e^{s_2t} & \cdots & c_{2n}e^{s_nt} \\ \vdots & & & \end{bmatrix}$$

assume s_1 has multiplicity m_1 and other have multiplicity 1.

$$e^{At} = \begin{bmatrix} c_{11}e^{s_1t} & c_{12}e^{s_1t} & \cdots & c_{1m_1}t^{(m_1-1)}e^{s_1t} & c_{1m_1+1}e^{s_2t} & \cdots \\ c_{21}e^{s_2t} & c_{22}e^{s_2t} & \cdots & c_{2m_1}t^{m_1-1}e^{s_1t} & c_{2m_1+1}e^{s_2t} & \cdots \\ & & & & & \end{bmatrix}$$

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Modes of operation:

If the initial state vector "namely x_0" is chosen equal or proportional to an eigenvalue or as linear combinations of some eigenvectors then $x(t)$ will include only the corresponding natural frequencies. In that case corresponding mode(s) are excited at the output.

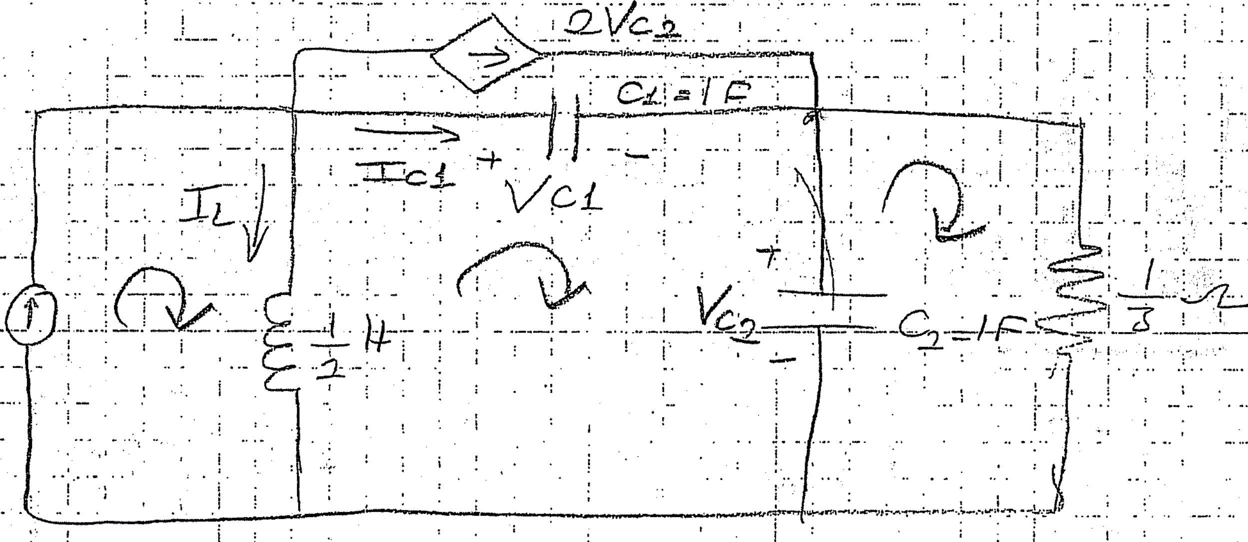
$s_1 = -1 \quad s_2 = -2 \quad s_3 = -3$

$x_1 \quad x_2 \quad x_3$ eigenvectors

modes

$$x_0 = k_1 x_1 \longrightarrow x(t) = k_2 x_0 e^{s_1 t}$$

Ex:



$$I_{C1} + 2V_{C2} = I_{C2} + I_R$$

$$C_1 \frac{dV_{C1}}{dt} + 2V_{C2} = \frac{V_{C2}}{\frac{1}{3}} + C_2 \frac{dV_{C2}}{dt}$$

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$$I_s = I_L + 2V_{c2} + I_{c1}$$ ✓

$$I_s = I_L + 2V_{c2} + C_1 \frac{dV_{c1}}{dt}$$ ✓

$$C_1 \frac{dV_{c1}}{dt} = I_s - I_L - 2V_{c2}$$ ✓

$$I_s - I_L - \cancel{2V_{c2}} + \cancel{2V_{c2}} = \frac{V_{c2}}{\frac{1}{3}} + C_2 \frac{dV_{c2}}{dt}$$ ✓

$$\boxed{\frac{dV_{c1}}{dt} = I_s - I_L - 2V_{c2}}$$ ✓

$$\boxed{\frac{dV_{c2}}{dt} = I_s - I_L - 3V_{c2}}$$ ✓

$$L \frac{dI_L}{dt} = V_L = V_{c1} + V_{c2}$$

$$\boxed{\frac{1}{2} \frac{dI_L}{dt} = V_{c1} + V_{c2}}$$

$$\begin{bmatrix} \dot{V}_{c1} \\ \dot{V}_{c2} \\ \dot{I}_L \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} V_{c1} \\ V_{c2} \\ I_L \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ 0 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} V_{c1} \\ V_{c2} \\ I_L \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} I_s$$

Let's find natural frequencies

$$\det(sI - A) = \begin{bmatrix} s & 2 & 1 \\ 0 & s+3 & 1 \\ -2 & -2 & s \end{bmatrix}$$

$$= s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 & -1 \\ 0 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\det(sI - A) = (s+1)(s^2+2s+2) = 0$$

$s_1 = -1 \quad s_2 = -1 + j \quad s_2 = -1 - j$

natural frequencies

Find the initial conditions which excites the complex modes only.

Find complex eigenvalues
conjugate

$A x_i = s_i x_i$

$(s_i I - A) x_i = \vec{0}$ → zero vector

$s_i I x_i = A x_i$

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$s = s_2$

$$(sI - A) = \begin{bmatrix} -1+j & 2 & 1 \\ 0 & 2+j & 1 \\ -2 & -2 & -1+j \end{bmatrix}$$

$$\begin{bmatrix} -1+j & 2 & 1 \\ 0 & 2+j & 1 \\ -2 & -2 & -1+j \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(-1+j)K_1 + 2K_2 + K_3 = 0$

$0 + (2+j)K_2 + K_3 = 0$

$-2K_1 - 2K_2 + (-1+j)K_3 = 0$

assume $K_3 = 1$

$K_2 = \frac{-1}{2+j}$ $\qquad (-1+j)K_1 + 2\left(\frac{-1}{2+j}\right) + 1 = 0$

$K_1 = -1.19 + j\,0.075$ $\qquad K_2 = \frac{-1}{2+j} \, (2-j) = \frac{-2+j}{5}$

$$x_2 = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} -1.19 + 0.075j \\ \frac{-2+j}{5} \\ 1 \end{bmatrix}$$

$\downarrow$

second mode related with $s_2 = -1+j$

$$X_3 = x_2^{*} = \begin{bmatrix} -1.19 - j\,0.075 \\ \frac{-2-j}{5} \\ 1 \end{bmatrix}$$

third mode corresponding to $s_3 = -1-j$

choose $x(0) = K_1 x_2 + K_2 x_3$

initial condition

$K_1 = 1$, $K_2 = 1$

$$x(0) = x_2 + x_3 = \begin{bmatrix} -1.19 \\ -2/5 \\ 1 \end{bmatrix}$$

in this case only effects of s_2 and s_3 will be observed at output

// K_1 ve K_2 için seçtiğimiz değerler bizim verdiğimiz değerlerdir ve istediğimiz aynı rakamı verebiliriz.

Complete Response:

$$\dot{x} = Ax + Bu$$

$$x(t) = x_{zi}(t) + x_{zs}(t)$$

zero-input zero-state

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For zero-input response:

$$\dot{x}_{zi}(t) = A\, x_{zi}(t)$$

$$x_{zi}(t) = x_0$$

For zero-state solution:

$$\dot{x}_{zs} = A\, x_{zs}(t) + B u(t)$$

$$x_{zs}(t_0) = 0$$

OR

$$x(t) = \underbrace{x_h(t)}_{\text{homogeneous}} + \underbrace{x_p(t)}_{\text{particular}}$$

let input $u(t) = \underbrace{P}_{\text{any number}} e^{s_0 t}$ (s_0 is not natural frequency of the system)

then $x_p(t) = \underbrace{K}_{\text{vector}}\, e^{s_0 t}$

put particular solution in diff. equation

$$\dot{x}(t) = Ax + Bu$$

$$s_0 K e^{s_0 t} = A K e^{s_0 t} + B P e^{s_0 t}$$

$$s_0 K e^{s_0 t} = (AK + BP) e^{s_0 t}$$

$$(s_0 I K - A K) e^{s_0 t} = B P e^{s_0 t}$$

$$(s_0 I - A) K \cancel{e^{s_0 t}} = B P \cancel{e^{s_0 t}}$$

$$\boxed{K = (s_0 I - A)^{-1} B P}$$

Sinusoidal Steady-State (SSS)

* Given a circuit, SSS condition exist

(a) $Re\{s_k\} < 0$ $s_k \rightarrow$ natural frequencies

(b) if $Re\{s_k\} \leq 0$ $s_k \neq j\omega$

where input $= A \cos(\omega t + \theta)$

* After checking if SSS exist or not solve the circuit in phasor domain

* In SSS the transient response is neglected.

* To find SSS we use phasors!

Phasor:

It is a complex number (#) representing the amplitude and phase. It can be in polar or rectangular form.

Polar:

$\underbrace{A}_{\text{amplitude}} e^{j\theta \rightarrow \text{phase}}$

Rectangular:

$A \cos\theta + j A \sin\theta$

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Ex: $V(t) = V_A \cos(\omega t + \theta)$

$= V_A \, Re\{e^{j(\omega t+\theta)}\}$

$= Re\{V_A \, e^{j(\omega t+\theta)}\} = Re\{V_A \, e^{j\theta} e^{j\omega t}\}$

$\rightarrow$ phasor

$V(t) \longrightarrow \bar{V}$

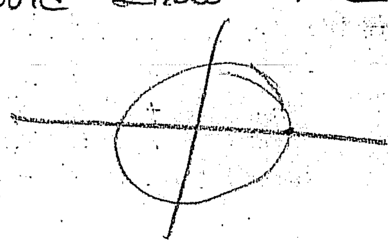
$V(t)$: signal, $\bar{V}$: phasor

$V(t) = V_A \cos(\omega t + \theta) \longrightarrow \bar{V} = V_A e^{j\theta}$

$= V_A(\cos\theta + j\sin\theta)$

* From $V(t)$ to $\bar{V}$ we neglect the information of frequency.

* But from $\bar{V}$ to $V(t)$ we should know the frequency information.



Ex: $V(t) = 2\sin(5t + 30^\circ)$

$= 2\sin(5t + 30^\circ) = 2\cos(5t + 30^\circ - 90^\circ)$

$= 2\cos(5t - 60^\circ)$

$\bar{V} = 2e^{-j60^\circ} = 2(\cos(-60^\circ) + j\sin(-60))$

$\bar{V} = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}j\right) = 1 - \sqrt{3}j$

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Properties of phasors:

① Additive

$$V(t) = V_1(t) + V_2(t) \longrightarrow \bar{V} = \bar{V_1} + \bar{V_2}$$

② Derivative

$$V(t) = Re\left\{ \bar{V} e^{j\omega t} \right\}$$

$$\frac{dV(t)}{dt} = \frac{d}{dt} Re\left\{ \bar{V} e^{j\omega t} \right\} = Re\left\{ \bar{V} \frac{d}{dt} e^{j\omega t} \right\}$$

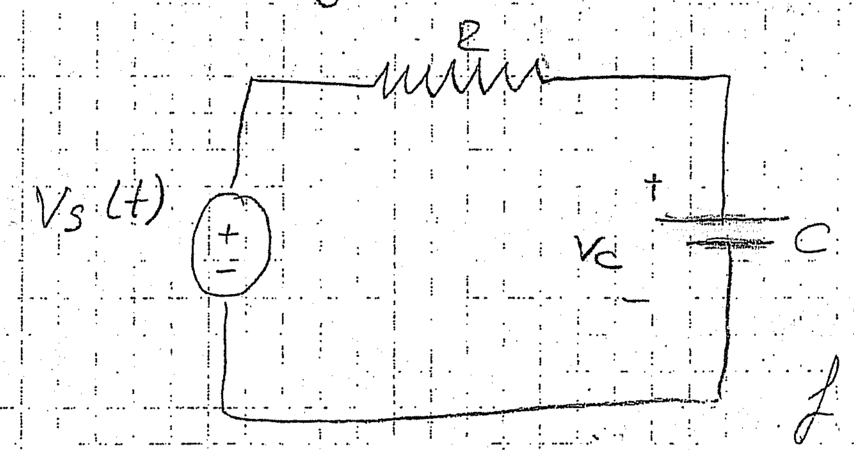
$$= Re\left\{ \bar{V} j\omega\, e^{j\omega t} \right\}$$

$$V(t) \longrightarrow \bar{V}$$

$$\frac{d}{dt} V(t) \longrightarrow \bar{V} j\omega$$

* We use phasors to find SSS solutions for circuits having bounded outputs

Ex:



$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} V_s$$

$V_s(t) = V_A \cos(\omega t)$, find $V_{c_{SSS}}$ when $t \geq 0$

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$$V_S(t) = V_A \cdot \cos(\omega t)$$
$$= V_A e^{j\omega t}$$

$$V_p(t) = Re\left\{ \overline{V_p}\, e^{j\omega t} \right\}$$

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} V_S$$

$$Re\left\{ V_A\, e^{j\omega t} \right\}$$

$$V_A \cos(\omega t)$$

$$Re\left\{ \overline{V_p}\, j\omega\, e^{j\omega t} \right\} + \frac{1}{RC} Re\left\{ \overline{V_p}\, e^{j\omega t} \right\} = \frac{1}{RC} Re\left\{ V_A\, e^{j\omega t} \right\}$$

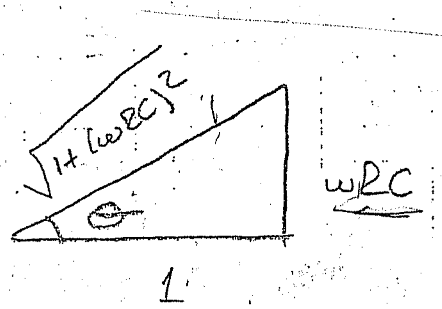
$$Re\left\{ \overline{V_p}\, j\omega\, e^{j\omega t} + \frac{1}{RC} \overline{V_p}\, e^{j\omega t} \right\} = Re\left\{ \frac{1}{RC} V_A\, e^{j\omega t} \right\}$$

$$\left(j\omega + \frac{1}{RC}\right) \overline{V_p} = \frac{1}{RC} V_A$$

$$\overline{V_p} = \frac{V_A}{j\omega RC + 1} = \frac{V_A (1 - j\omega RC)}{1 + (\omega RC)^2} = \frac{V_A (1 - j\omega RC)}{\sqrt{1 + (\omega RC)^2}\sqrt{1 + (\omega RC)^2}}$$

$$(1 - j\omega RC)$$

$$\overline{V_p} = \frac{V_A}{\sqrt{1 + (\omega RC)^2}} \left[- \frac{j\omega RC}{\sqrt{1 + (\omega RC)^2}} + \frac{1}{\sqrt{1 + (\omega RC)^2}} \right]$$



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$$\theta = \tan^{-1}\left(\frac{-\omega RC}{1}\right)$$

$$\bar{V_p} = \underbrace{\frac{V_A}{\sqrt{1+(\omega RC)^2}}}_{\text{amplitude}} e^{j\theta} \rightarrow \text{phase}$$

$$V_p(t) = Re\left\{\bar{V_p}\, e^{j\omega t}\right\} = Re\left\{\frac{V_A}{\sqrt{1+(\omega RC)^2}}\, e^{j(\omega t+\theta)}\right\}$$

$$V(t) = \frac{V_A}{\sqrt{1+(\omega RC)^2}} \cos(\omega t + \theta)$$

$$V_p(t) = \frac{V_A}{\sqrt{1+(\omega RC)^2}} \cos(\omega t + \tan^{-1}(-\omega RC))$$

→ this solution is the particular solution it is also equal to $V_{css}(t)$

Circuit Analysis in Phasor Domain:

KVL, KCL → can be used

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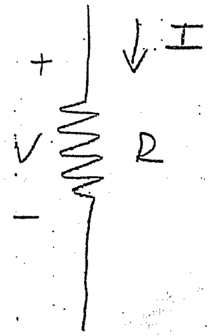
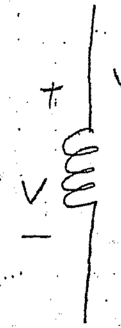
Devices



$V = R.I$

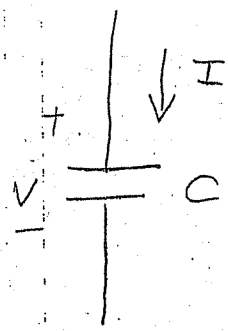
$$\boxed{\bar V = R.\bar I}$$



$$L\frac{dI}{dt} = V_L$$

$$\boxed{L(j\omega)\bar I = \bar V_L}$$

phasor domain!



$$C\frac{dV}{dt} = I$$

$$\boxed{Cj\omega\bar V = \bar I}$$

özdirenç　　　　giriş lanı

Impedance and Admittance:

Impedance "$Z(j\omega)$" at any angular frequency "ω" is the ratio of phasor of $\bar V$ to phasor of $\bar I$ in a two terminal element.

$$\underset{\text{impedance}}{Z(j\omega)} = \frac{\bar V}{\bar I} = \underbrace{\left|\frac{\bar V}{\bar I}\right|}_{\text{amplitude}} e^{j\angle Z(j\omega)}$$

phase

Ohms

$$Z(j\omega) = \frac{\bar V}{\bar I}$$

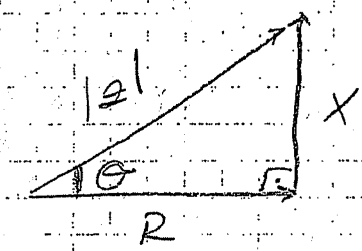
$$\text{Admittance} = \frac{1}{Z(j\omega)} = Y(j\omega) \quad \text{siemens}$$

$$Z = R(j\omega) + jX(j\omega)$$

R = resistance → tepki

X = reactance

$$|Z| = \sqrt{R^2 + X^2}$$



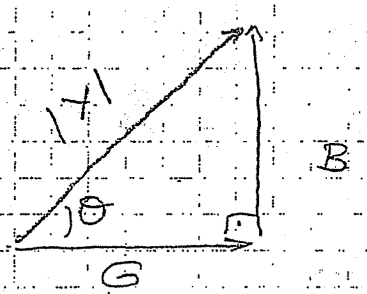
$$\angle Z(j\omega) = \theta$$

Admittance $\quad Y(j\omega) = \dfrac{1}{Z(j\omega)} = G + jB$

G = conductance → iletkenlik

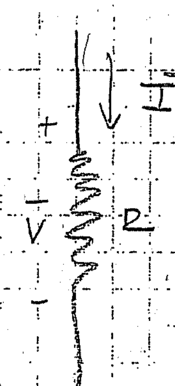
B = susceptance – sanal geçir

$$|Y| = \sqrt{G^2 + B^2}$$



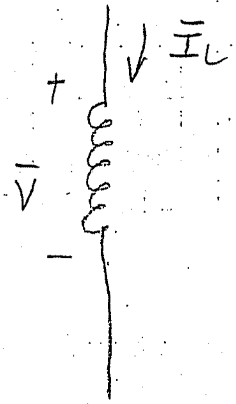
$$\angle Y(j\omega) = \theta$$

Impedance / Admittance of R, L, C



$$Z(j\omega) = R$$

$$Y(j\omega) = \frac{1}{Z(j\omega)} = \frac{1}{R} = G$$



$$\bar{V} = j\omega L \bar{I}$$

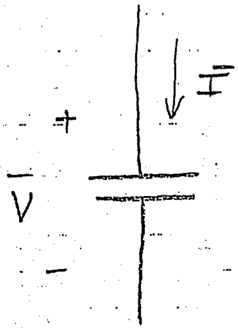
$$Z(j\omega) = j\omega L$$

$$Y(j\omega) = \frac{1}{j\omega L}$$

$$V_L = L \cdot \frac{dI_L}{dt}$$

$$\bar{V}_L = j\omega \cdot L \cdot \bar{I}_L$$

$$\bar{I}_L = \frac{\bar{V}_L}{j\omega L}$$



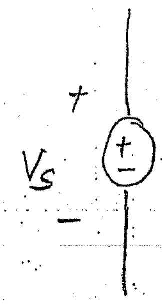
$$\bar{V} = \frac{\bar{I}}{j\omega C}$$

$$Y(j\omega) = j\omega C$$

$$Z(j\omega) = \frac{1}{j\omega C}$$

$$C\frac{dV}{dt} = I \xrightarrow{\text{phasor}} j\omega C \bar{V} = \bar{I}$$

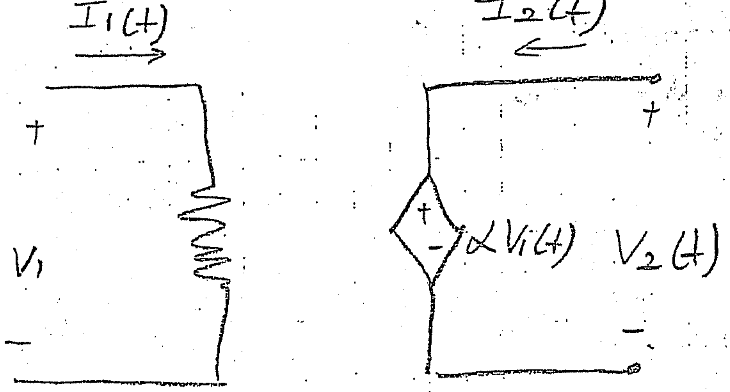
Source



$$V_s(t) = V_m \cos(\omega t + \phi_s)$$

$$\downarrow \text{phasor}$$

$$\boxed{\bar{V}_s = V_m e^{j\phi_s}}$$



$$V_2(t) = \alpha V_1(t)$$

$$\bar{V}_2 = \alpha \bar{V}_1$$

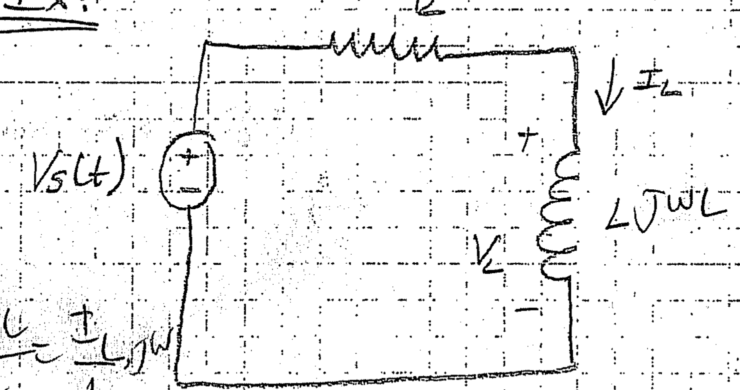
Phasor domain circuit analysis: *(only sinusoidal steady-state values are obtained; no transient part is evaluated)

(1) Turn all circuit parameters, variables, coefficients to phasor domain.

(2) Use KVL, KCL, mesh, node analysis metods.

(3) After obtaining the result (signal); turn back into time domain.

Ex:



$L\frac{dV_L}{dt} = \frac{I_L}{L} j\omega$

$V_s(t) = V_m \cos(\omega t + \phi_V)$

Does SSS exist?

Yes.

$s + \frac{R}{L} = 0$

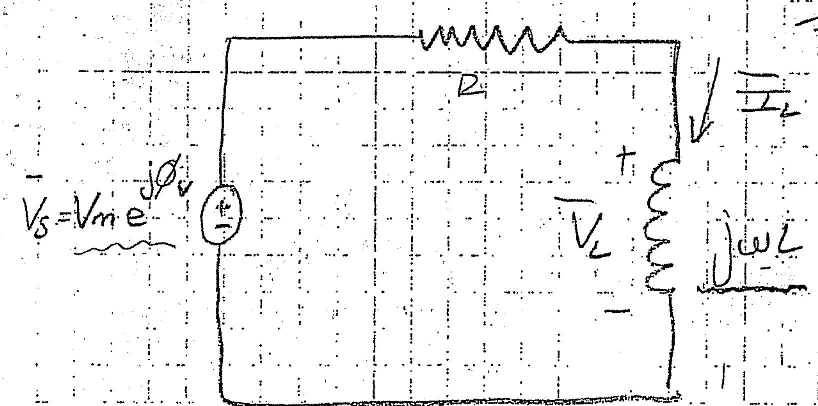
$s_1 = -\frac{R}{L} \neq j\omega$

Find SSS response for $I_L(t)$?

$-V_s + V_R + V_L = 0$

$V_R + V_L = V_s$

Transform into phasor domain

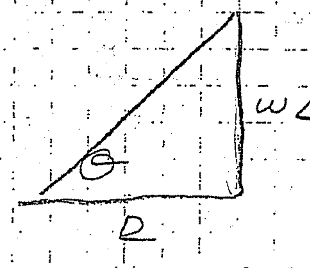


$\overline{I_L} = \frac{\overline{V_L}}{j\omega L}$

$$\overline{I_L} = \frac{\overline{V_s}}{R + j\omega L}$$

$$\overline{I_L} = \frac{V_m e^{j\phi_V}}{R + j\omega L}$$

$$\overline{I_L} = \frac{V_m e^{j\phi_V}}{\sqrt{R^2 + (\omega L)^2}\, e^{j\tan^{-1}\left(\frac{\omega L}{R}\right)}}$$



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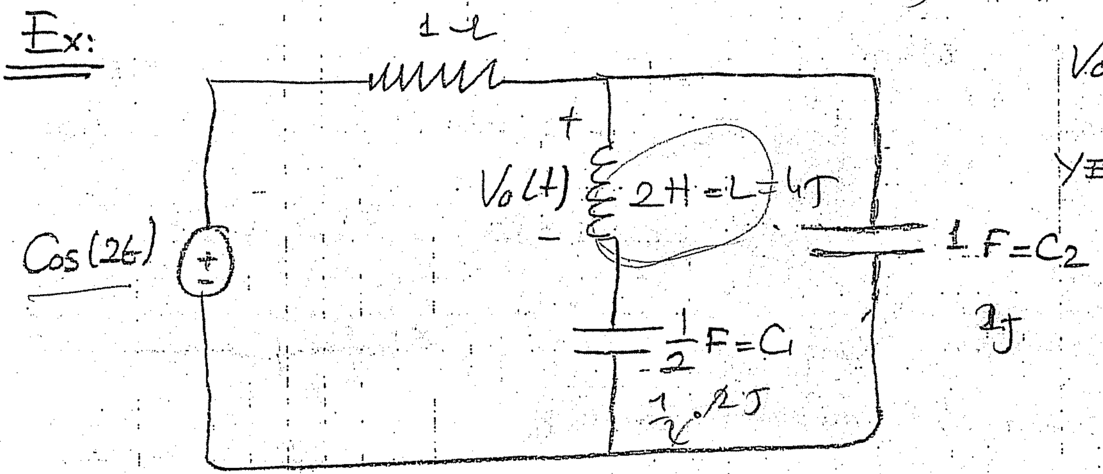
$$\overline{I}_L = \frac{V_m}{\sqrt{R^2+(\omega L)^2}} e^{j\left(\phi_v - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)}$$

back to time domain

$$I_L(t) = \frac{V_m}{\sqrt{R^2+(\omega L)^2}} \cos\left(\omega t + \phi_v - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

Ex:

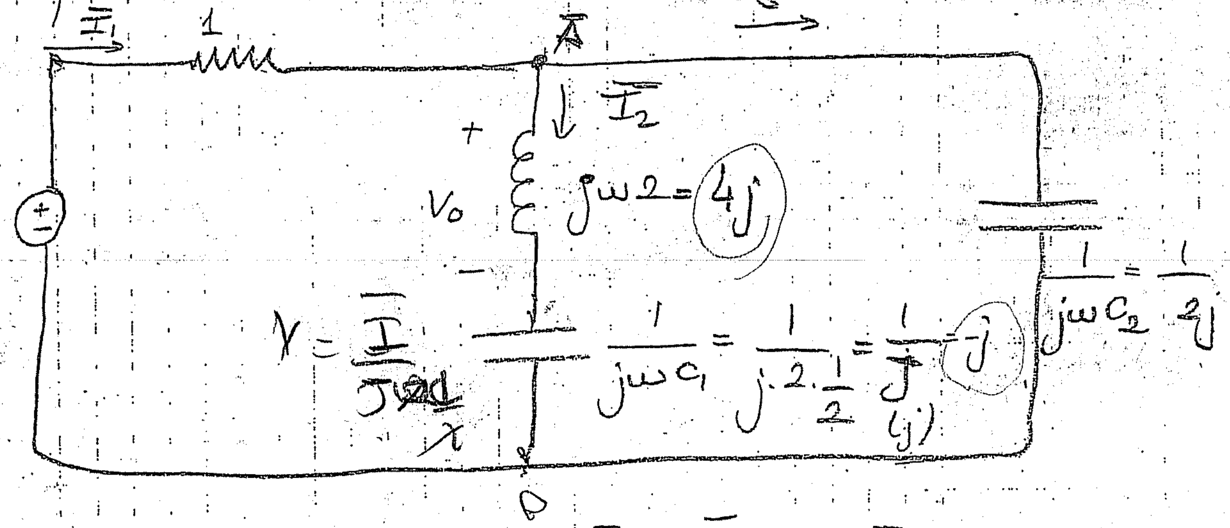


$V_o(t) \rightarrow$ SSS

YES SSS exist

$\overline{V} = j\omega \overline{I}$
$2 \cdot 2 j$

Turn into phasor domain: $(\omega = 2)$



$$\overline{I_1} = \overline{I_2} + \overline{I_3}$$

$$1 - \overline{A} = \frac{\overline{A}}{3j} + 2j\overline{A}$$

$$\frac{1-\overline{A}}{1} = \frac{\overline{A}}{4j+(-j)} + \frac{\overline{A}}{\frac{1}{2j}}$$

$$3j - 3j\overline{A} = \overline{A} - 6\overline{A}$$

$$\overline{A} = \frac{3j}{3j-5} = \frac{-9+15j}{-34} = \frac{9-15j}{34}$$

$$1 - \overline{A} = \frac{j\overline{A}}{-j} + 2j A \ldots \frac{1}{j} \quad \frac{-j}{j}$$

$I = \frac{A}{(4j)(-j)}$ $\quad \bar{V}_0 = j\overset{4}{\omega L}\frac{A}{4j+(-j)}$

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$$\bar{V}_0 = \bar{A} * \frac{4j}{4j+(-j)} \Rightarrow \bar{V}_0 = \frac{4}{3}\bar{A}$$

$$\bar{V}_0 = \frac{4}{3}\left(\frac{9-15j}{34}\right)$$

$$\bar{V}_0 = \underbrace{0.68}_{\text{magnitude}} e^{j\overbrace{59.4^\circ}^{\to\ \text{phase}}}$$

$$\left(\frac{12}{34} - \frac{10j}{12}\right)$$

A ⟷ Y Transformation:

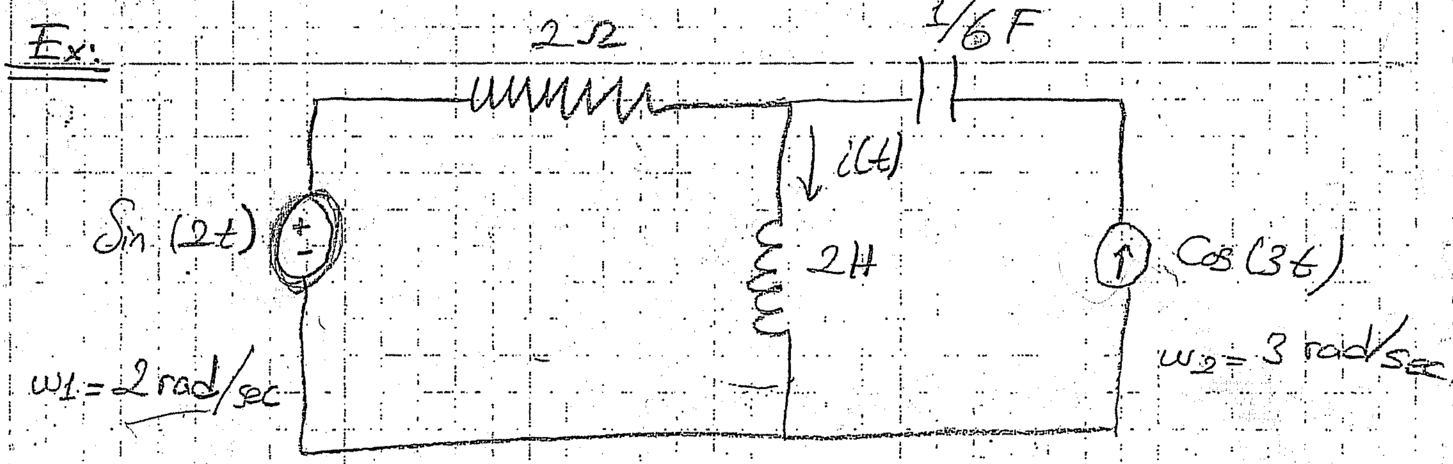
Superposition:

* Superposition is always applicable in time domain

* ① If the sources have same input frequency, superposition can be applied in phasor domain.

② If the sources have different input frequency, superposition can not be applied in phasor domain. Instead, time domain results of phasors should be added.

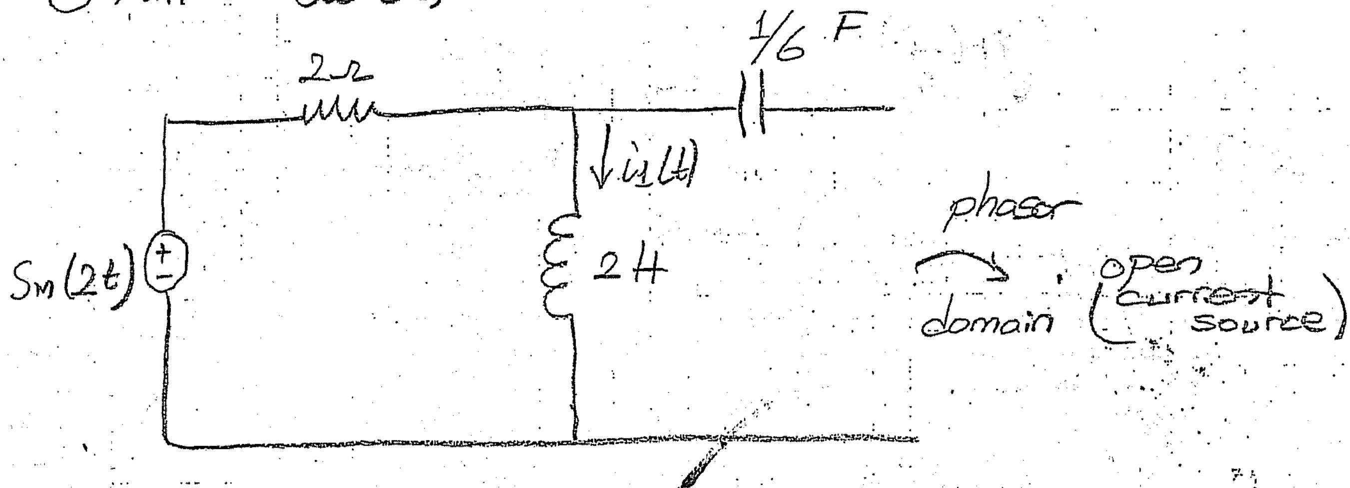
Ex:



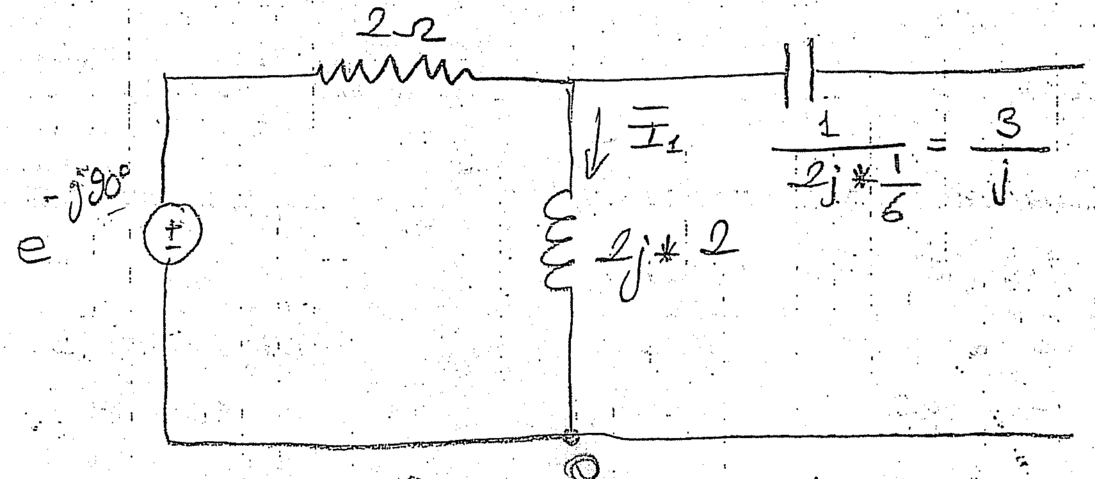
Find $i(t)$ in SSS

Apply superposition

① Kill $\cos(3t)$



$\sin(2t) = \cos(2t - 90°)$



$$\bar{I_1} = \frac{e^{-j90°}}{2+4j} = \frac{-j}{2+4j}(2-4j) = \frac{-4-2j}{20} = -\left(\frac{1}{5} + \frac{1}{10}j\right)$$

$$I_1 = -\left(\frac{1}{5} + \frac{1}{10}j\right) \Rightarrow I(t) = -\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{1}{10}\right)^2} \cos\left(2t + \tan^{-1}\frac{1/10}{1/5}\right)$$

$$I_1(t) = -0.374 \cos\left(2t + \tan^{-1}\frac{1}{2}\right)$$

$$I_1(t) = -0.374 \cos(2t + 26.56°)$$

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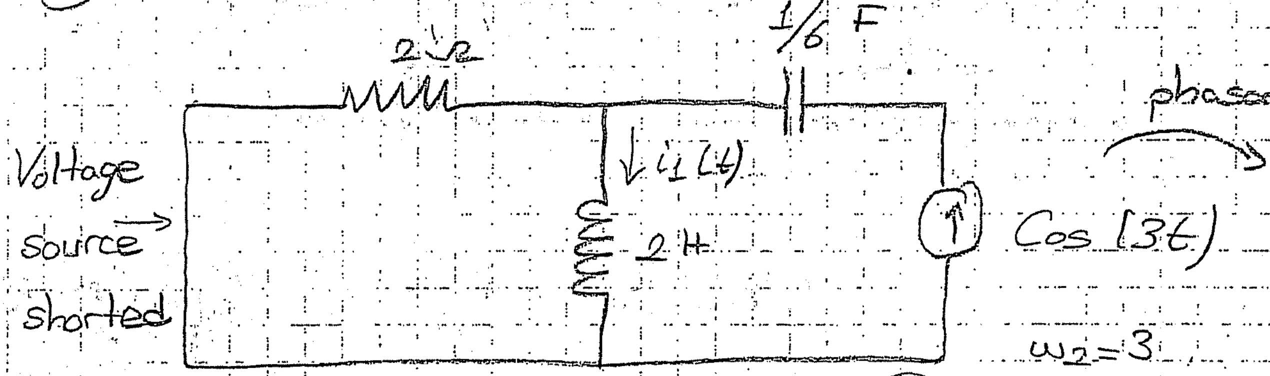
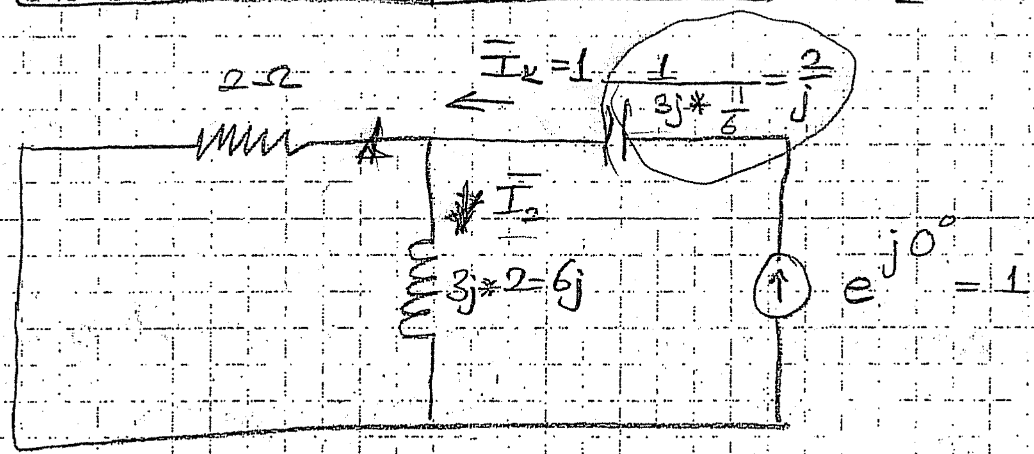
(2) Kill Sin(2t)





$$\bar{I}_2 = \frac{2 * \bar{I}_L}{2 + 6j} = \frac{2 * 1}{2 + 6j} = \frac{2}{2 + 6j} = \frac{2(2-6j)}{40}$$

$(2-6j)$

$$= \frac{1}{10} - \frac{3}{10}j$$

$$\Rightarrow I_2(t) = \sqrt{\left(\frac{1}{10}\right)^2 + \left(-\frac{3}{10}\right)^2} \cos\left(3t + \tan^{-1}\frac{-3/10}{\frac{1}{10}}\right)$$

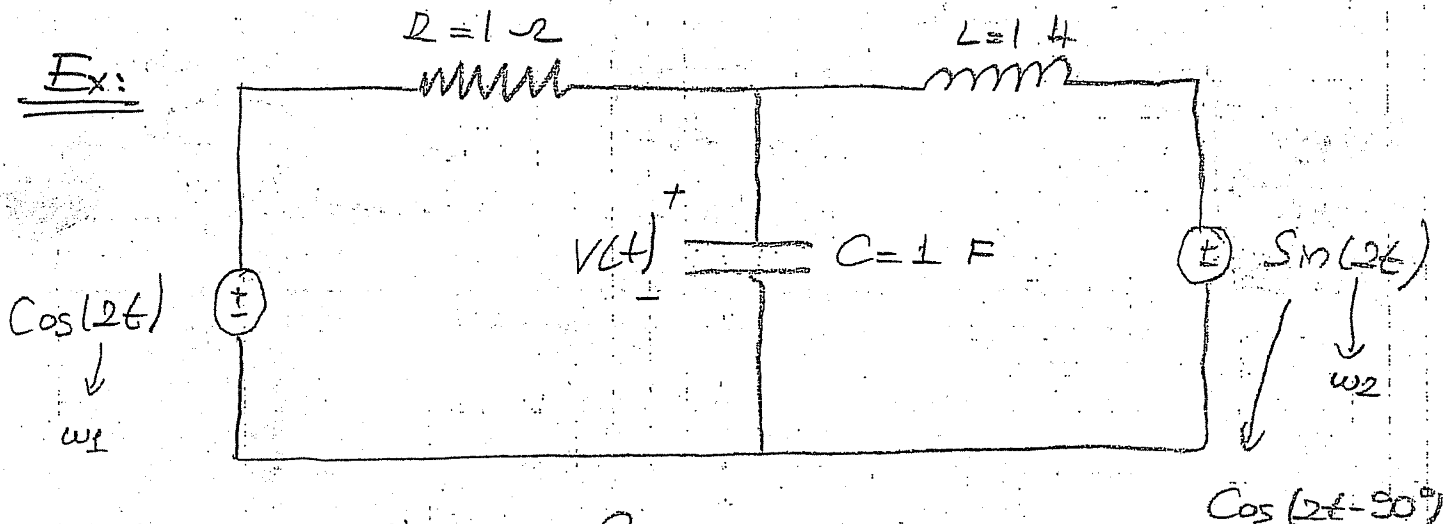
$$I_2(t) = 0.44 \cos\left(3t + \tan^{-1}(-3)\right)$$

$$I_2(t) = 0.44 \cos(3t - 71.56^\circ)$$

$I(t) = I_1(t) + I_2(t)$

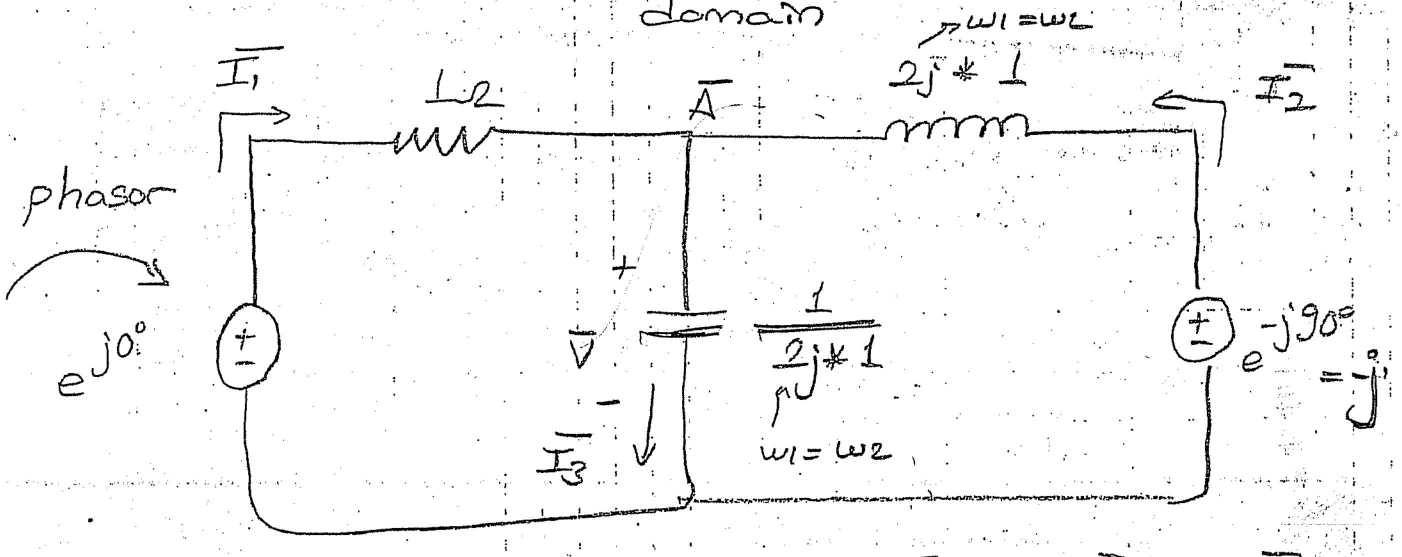
$I(t) = -0.374 \cos(2t + 26.56°) + 0.44 \cos(3t - 71.56°)$

Ex:



What is V(t) SSS?

$\omega_1 = \omega_2 = 2$ rad/sec apply superposition in phasor domain



$\bar{V} = \bar{A}$

$\bar{I_1} + \bar{I_2} = \bar{I_3}$

$\frac{1 - \bar{A}}{1} + \frac{-j - \bar{A}}{2j} = \frac{\bar{A}}{\frac{1}{2j}}$

$2j - 2j\bar{A} - j - \bar{A} = -4\bar{A}$

$j = 2j\bar{A} - 3\bar{A}$

$j = (2j - 3)\bar{A}$

$\bar{A} = \frac{j}{2j - 3} \frac{(2j+3)}{} = \frac{2 - 3j}{13}$

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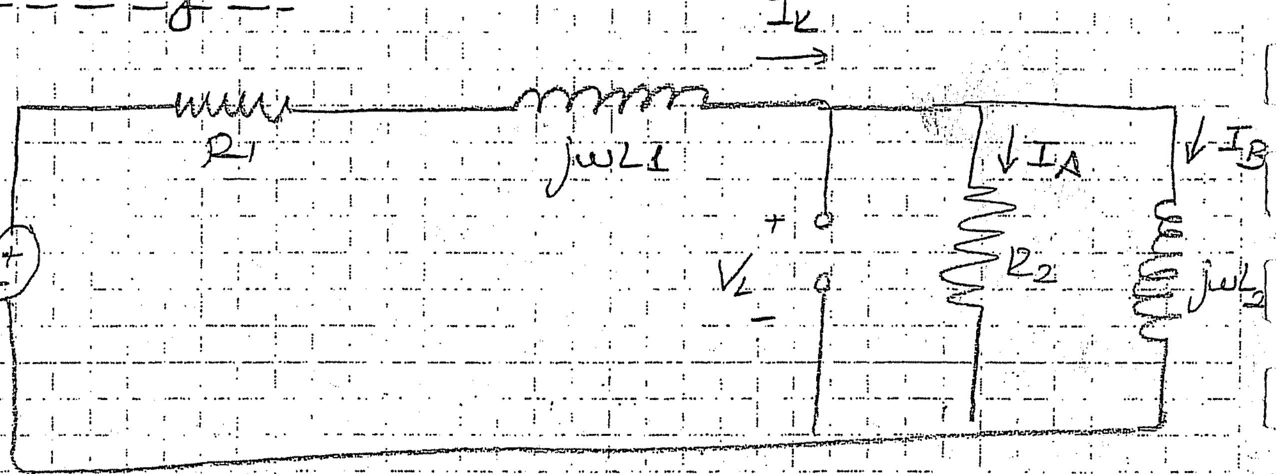
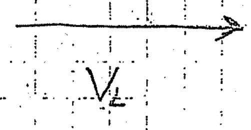
$$A = V(t) = \sqrt{\left(\frac{2}{13}\right)^2 + \left(\frac{-3}{13}\right)^2} \, \cos\left(2t + \arctan\left(\frac{-3/13}{2/13}\right)\right)$$

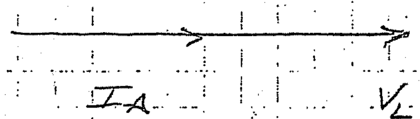
Phasor Diagram:



① Choose one of the phasors as reference (V_L)

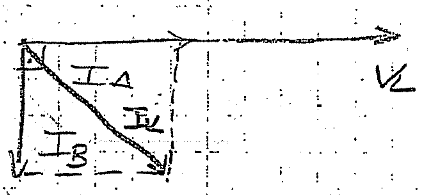


② I_A is with the same direction with V_L



$$|I_A| = \frac{|V_L|}{R_2}$$

③



I_B lags V_L by 90° since it is the current over the inductor

$$|I_B| = \frac{|V_L|}{\omega L_2}$$

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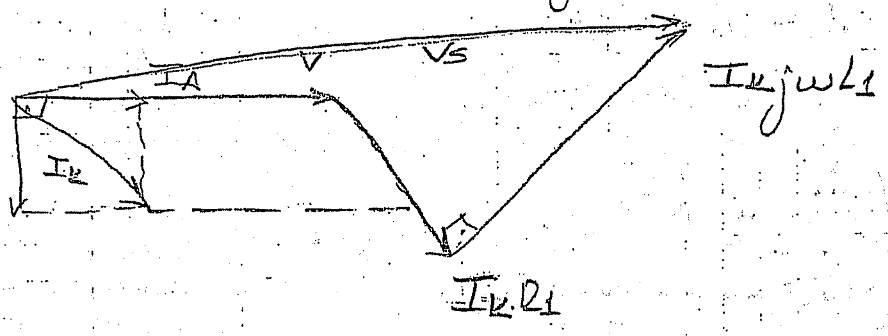
(4) $I_L = I_A + I_B$

(5) $V_s = V_L + I_L(R_1 + j\omega L_1)$



Sinusoidal Steady-State Power

Instantaneous Power: (Power at any instant)

$$P(t) = V(t)\,I(t) \quad (\text{Watts})$$

Average Power: (It is associated with periodic signals)
It is the average of instantaneous power at one period

real power

$$P = P_{avg} = \frac{1}{T}\int_{k}^{k+T} P(t)\,dt \quad (\text{Watts})$$

Complex Power: It is the complex sum of average power and reactive power

power due to energy storage

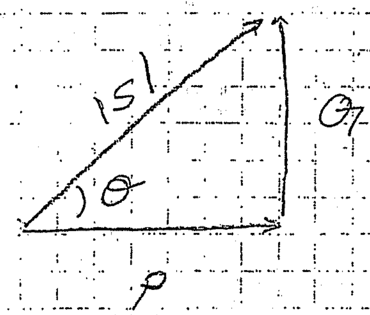
$$S = P + jQ$$

reactive power (VAR) (Volt-amper-reactive)

complex power (VA) (volt-amper)

average power (W) (Watts)

$$|S| = \sqrt{P^2 + Q^2}$$



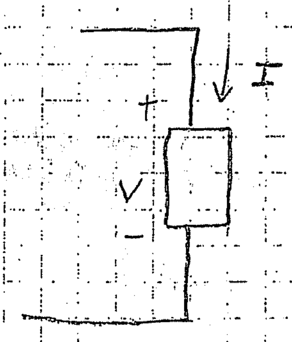
→ apparent power (VA)

$$\theta = \theta_v - \theta_i$$

θ_v → angle of voltage

θ_i → angle of current

θ → power factor angle



$$V(t) = V_m \cos(\omega t + \theta_v)$$

$$I(t) = I_m \cos(\omega t + \theta_i)$$

$$\text{power factor} = pf = \cos\theta = \frac{P}{|S|}$$

$$\text{reactive factor} = rf = \sin\theta = \frac{Q}{|S|}$$

RMS values

$V(t) = a \sin(\omega t) \longrightarrow V_{rms} = \frac{a}{\sqrt{2}}$

$V(t) = a \, Tri(\omega t) \longrightarrow V_{rms} = \frac{a}{\sqrt{3}}$

$V(t) = a \, Sqr(\omega t) \longrightarrow V_{rms} = a$

$$f_{rms} = \sqrt{\frac{1}{T} \int_0^T f^2(t)\, dt}$$

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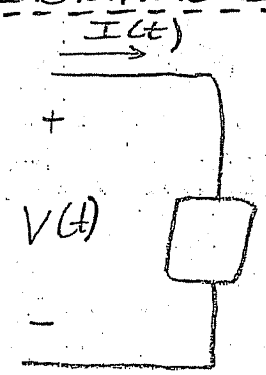
Instantaneous Power:



$$V(t) = V_m \cos(\omega t + \theta_v)$$
$$I(t) = I_m \cos(\omega t + \theta_i)$$

assume $I(t)$ is reference signal

(shift signals $-\theta_i$ units)

$$V(t) = V_m \cos(\omega t + \theta_v - \theta_i)$$
$$I(t) = I_m \cos(\omega t)$$

$$P(t) = V(t) \cdot I(t) = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos(\omega t)$$

$$\cos(A) \cdot \cos(B) = \frac{1}{2} \cos(A+B) + \frac{1}{2} \cos(A-B)$$

$$\Rightarrow P(t) = V_m I_m \left[\frac{1}{2} \cos(2\omega t + \theta_v - \theta_i) + \frac{1}{2} \cos(\theta_v - \theta_i) \right]$$

$$P(t) = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(2\omega t + \theta_v - \theta_i)$$

$$\cos(a+b) = \cos a \cdot \cos b - \sin a \sin b$$

$$\Rightarrow P(t) = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \Big[\cos(2\omega t) \cos(\theta_v - \theta_i) - \sin(2\omega t) \sin(\theta_v - \theta_i) \Big]$$

$$P(t) = \underbrace{\frac{V_m I_m}{2} \cos(\theta_v - \theta_i)}_{P} + \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \cos(2\omega t)$$

$$- \underbrace{\frac{V_m I_m}{2} \sin(\theta_v - \theta_i)}_{Q} \sin(2\omega t)$$

$$P(t) = P + P \cdot \cos(2\omega t) - Q \sin(2\omega t)$$

$$P_{avg} = \frac{1}{T} \int_0^T P(t)\, dt = \frac{1}{T} \int_0^T \left[P + P\cos(2\omega t) - Q \sin(2\omega t) \right] dt$$

$$P_{avg} = \frac{1}{T} \left[Pt \Big|_0^T + P\left(-\frac{\sin 2\omega t}{2\omega} \right) \Big|_0^T - Q \frac{\cos 2\omega t}{2\omega} \Big|_0^T \right]$$

$$P_{avg} = \frac{1}{T} \left[PT + P \frac{1}{2\omega} \left[\underbrace{-\sin(2\omega T) + \sin(0)}_{0} \right] - Q \frac{1}{2\omega} \left[\underbrace{\cos(2\omega t) - \cos(0)}_{0} \right] \right]$$

$$P_{avg} = P$$

<u>Energy:</u>

$$W(t) = \int_0^t P(t')\, dt' \qquad P(t) = \frac{dW(t)}{dt}$$

$$t = t_1 = \frac{2\pi}{2\omega} \quad (\text{period of power})$$

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$$W(T_1) = \int_0^{T_1} \left(P + P\cos(2\omega t') - Q_r \sin(2\omega t') \right) dt'$$

$$W(T_1) = \int_0^{T_1} P\, dt' + \underbrace{\int_0^{T_1} P\cos(2\omega t')\,dt'}_{0} + \underbrace{\int_0^{T_1} Q_r \sin(2\omega t')\,dt'}_{0}$$

$$W(T_1) = P T_1$$

* Q_r is the reactive power due to inductor or capacitor. Inductor or capacitor do not deliver a net amount of energy to the circuit.

Observations:

① $P_{avg} = P = \dfrac{V_m I_m}{2} \cos(\theta_v - \theta_i)$

② The frequency of $P(t)$ (instantaneous power) is twice the frequency of voltage or current.

Ex: $\theta_v = \theta_i$ (purely resistive circuit)

$$I_R = I_m \cos(\omega t)$$

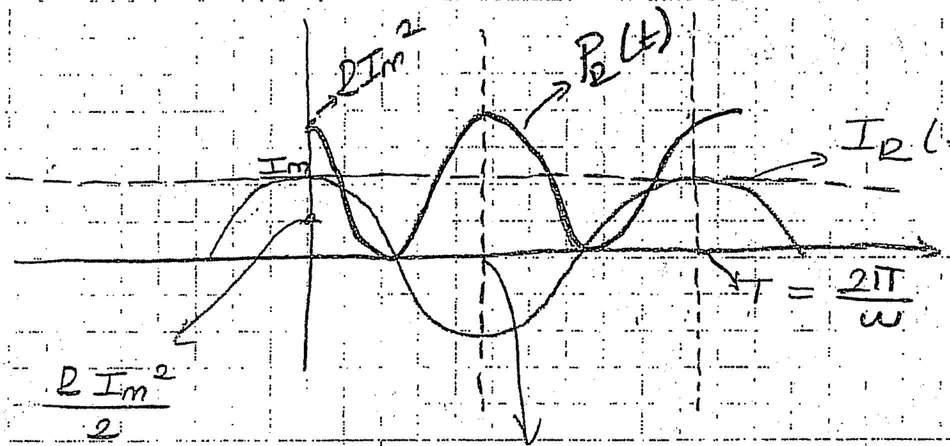
$$V_R = R \cdot I_m \cos(\omega t)$$

$$P_R(t) = V_R \cdot I_R = R I_m^2 \cos(\omega t)\cos(\omega t)$$

↑ instantaneous power

$$P_R(t) = R I_m^2 \left[\frac{1}{2}\cos(2\omega t) + \frac{1}{2}\cos(\omega t - \omega t) \right]$$

$$P_R(t) = \frac{R\,I_m^2}{2}\left[1 + \cos(\underbrace{2\omega t})\right]$$ angular frequency of power



$\omega = 2\pi f$

$T_{power} = \frac{2\pi}{2\omega}$

(3) For purely resistive element

$\theta_v = \theta_i \qquad P(t) = P + P\cos(2\omega t)$

$P(t) \geq 0$

(4) For purely capacitor element C:

$\theta_i - \theta_v = 90^\circ$

$V(t) = V_m \cos(\omega t) \longrightarrow T \qquad T = \frac{1}{f} = \frac{2\pi}{\omega}$

$$C\frac{dV(t)}{dt} = I(t) = \underbrace{C\,V_m\,\omega}_{I_m}\left(-\sin(\omega t)\right)$$

$$P(t) = V_m\cos(\omega t)\left[C V_m \omega\,(-\sin(\omega t))\right]$$

$$= V_m^2 C\omega\left[\cos(\omega t)\sin(\omega t)\right]\frac{2}{2}$$

$$= -\frac{V_m^2 C\omega}{2}\sin(2\omega t)$$

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$$P(t) = -\frac{I_m V_m}{2} \sin(2\omega t) \rightarrow T_k = \frac{T}{2} = \frac{\pi}{\omega}$$

↳ instantaneous power

$$P_{avg} = \frac{1}{T}\int_0^T p(t)\,dt = 0$$

$$Q_1 = -\frac{\omega C V_m^2}{2} \quad (VAR)$$

$$W_{CA} = \frac{1}{2} C (V_{rms})^2 = \frac{1}{2} C \left(\frac{V_m}{\sqrt{2}}\right)^2$$

↓ Average capacitor energy

$$Q_2 = -2\omega W_{CA}$$

5 For purely inductive element L:

$$\theta_v - \theta_i = 90^\circ$$

$$I(t) = I_m \sin(\omega t)$$

$$V(t) = L \frac{dI(t)}{dt} = \overbrace{L\,I_m\,.\omega}^{V_m} \cos(\omega t)$$

$$P(t) = L\omega I_m^2 \sin(\omega t)\cos(\omega t) \cdot \frac{2}{2}$$

$$P(t) = \frac{V_m I_m}{2} \sin(2\omega t) = \frac{L\omega I_m^2}{2} \sin(2\omega t)$$

$$P_{avg} = \frac{1}{T}\int_0^T P(t)\,dt = 0$$

$$Q_r = \frac{\omega L I_m^2}{2} \quad (VAR)$$

$$W_{LA} = \frac{1}{2} L (I_{rms})^2 = \frac{1}{2} L \left(\frac{I_m}{\sqrt{2}}\right)^2$$

average inductor energy

$$Q_r = 2\omega W_{LA}$$

The power associated with pure inductive or capacitive circuit is referred as reactive power Q_r.

$$Q_r = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i)$$

$$I(t) = I_m \cos(\omega t + \theta_i)$$

$$V(t) = V_m \cos(\omega t + \theta_v)$$

for inductive elements $0 < \theta_v - \theta_i < 90°$ $Q_r > 0$

for capacitive elements $-90° < \theta_v - \theta_i < 0$ $Q_r < 0$

Power factor $= pf = \cos(\theta_v - \theta_i)$

$$\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$$

if $\theta_v - \theta_i > 0$ (inductive) lagging power factor.

current lags voltage for inductive circuits

power factor $= \cos(\theta_v - \theta_i)$ if $\theta_v - \theta_i < 0$ (capacitive) leading power factor.

current leads voltage for capacitive circuits.

Ex: $V(t) = 100 \cos(\omega t + 15^\circ)$ Volts

$I(t) = 4 \sin(\omega t - 15^\circ)$ Amps $4\cos(\omega t - 15 - 90)$
$(\omega t - 105)$

$P = ?$, $Q = ?$

$\theta_v = 15^\circ$

$P = P_{avg} = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$ $\theta_i = -105^\circ$

$I(t) = 4 \cos(\omega t - 15^\circ - 90^\circ) = 4 \cos(\omega t - 105^\circ)$

$P = \frac{100 \cdot 4}{2} \cos(15 - (-105))$

$P = 200 \cos(120^\circ) = 200 * \left(-\frac{1}{2}\right) = -100$ Watts (source)

P (−) ⟶ delivering power (source)
P (+) ⟶ using power (resistive element)

$Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) = \frac{100 \cdot 4}{2} \sin(120^\circ) = 200 \frac{\sqrt{3}}{2}$

$Q = 173.21$ VAR (inductive since Q is positive)

Complex Power:

$S = P + jQ$ (VA)

S ⟶ complex; P ⟶ average; Q ⟶ reactive)

$$\boxed{S = \frac{V_m I_m}{2} \cos(\underbrace{\theta_v - \theta_i}_{\theta}) + j \frac{V_m I_m}{2} \sin(\theta_v - \theta_i)}$$

$$S = \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)}$$

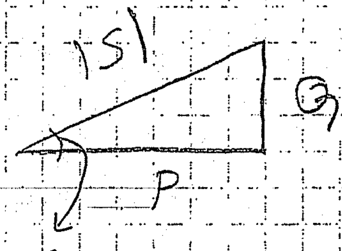
$$S = \frac{1}{2} \underbrace{V_m e^{j(\theta_v)}}_{\text{phasor of voltage}} \underbrace{I_m e^{-j(\theta_i)}}_{\text{conjugate of phasor of current}}$$

$$S = \frac{1}{2} V I^*$$

$V \longrightarrow$ voltage phasor

$I \longrightarrow$ current phasor

$$S = V_{rms} I_{rms}^* \qquad \left(V_{rms} = \frac{V}{\sqrt{2}} \qquad I_{rms} = \frac{I}{\sqrt{2}} \right)$$



$$|S| = \sqrt{Q^2 + P^2}$$

$$|S| = |V_{rms}| \, |I_{rms}^*|$$

$$\theta = \theta_v - \theta_i$$

$$|I_{rms}| = |I_{rms}^*|$$

$$\tan\theta = \frac{Q}{P} \qquad pf = \cos\theta = \cos(\theta_v - \theta_i)$$

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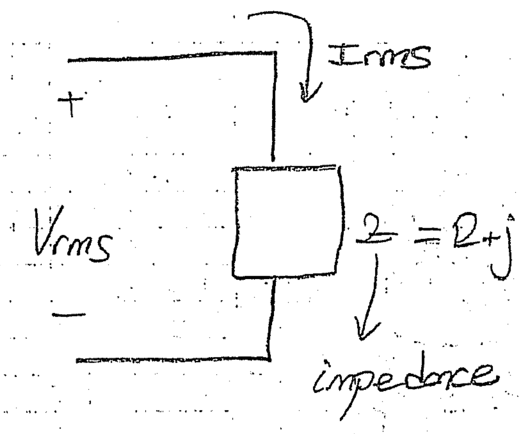

$$S = V_{rms} \, I_{rms}^*$$

$$I_{rms} = \frac{V_{rms}}{Z}$$

$$S = V_{rms} \left(\frac{V_{rms}}{Z} \right)^* = V_{rms} \frac{V_{rms}^*}{Z^*}$$

$$S = \frac{|V_{rms}|^2}{Z^*}$$

$$S = \frac{|V_{rms}|^2}{Z^*} \cdot \frac{Z}{Z} = Z \frac{|V_{rms}|^2}{|Z|^2} = Z \, |I_{rms}|^2$$

$$S = \frac{|V_{rms}|^2 Z}{|Z|^2} = |V_{rms}|^2 \frac{R}{R^2 + X^2} + j \, |V_{rms}|^2 \frac{X}{R^2 + X^2}$$

$$\text{power factor} = \cos(\underbrace{\theta_v - \theta_i}_{\theta})$$

$\theta = 0 \Rightarrow$ circuit is purely resistive

$\theta = \mp 90 \Rightarrow$ circuit is purely (reactive)

(+90 purely inductive
−90 purely capacitive)

$p(t) \longrightarrow$ instantaneous power

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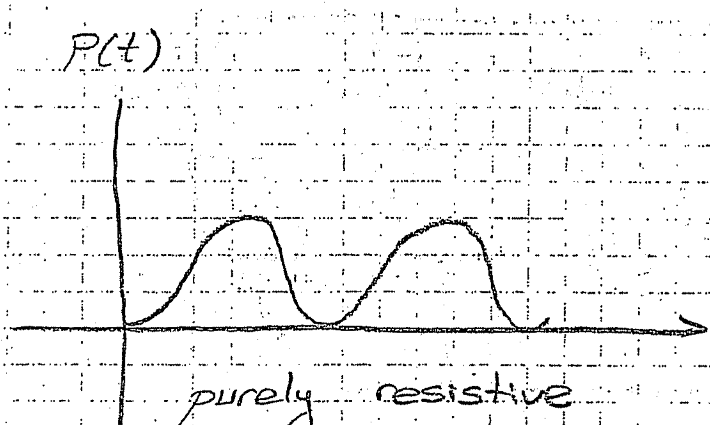
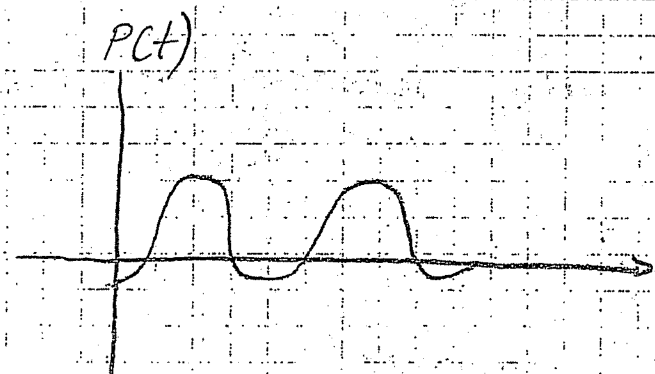
$\cos\theta = 1 \quad \theta = 0$ resistive



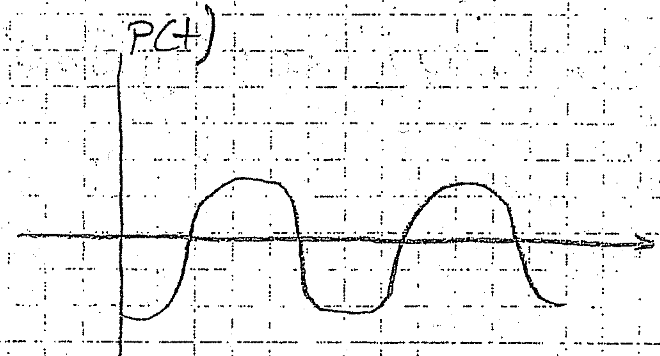
purely resistive



$\cos\theta = 0.8$

$0 < \cos(\theta) < 1$

resistance + reactance

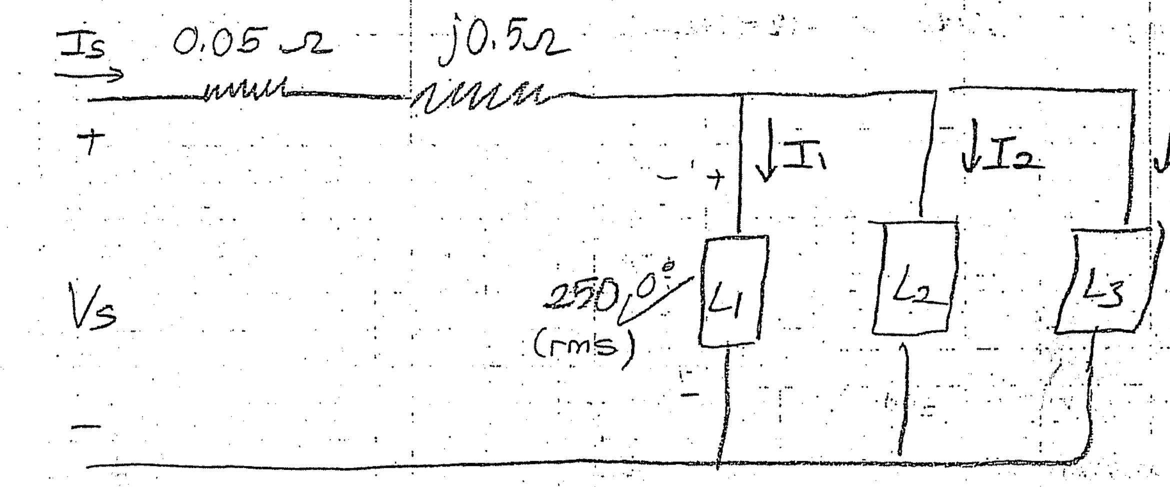


$\cos\theta = 0$

$\theta = \mp 90$ → reactive (capacity, inductive)

* Reactive power is a measure of energy between the source and the circuit without being used by the circuit.

Ex:



Load 1: uses 8 kW with lagging of 0.8
Load 2: absorbs 20 kVA with leading of 0.6
Load 3: $Z = 2.5 + j5.0\ \Omega$

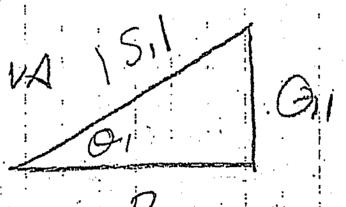
$f = 60$ Hz (for V_s) $V_s = ?$

$$V_s = I_s\,[0.05 + j0.5] + 250\angle 0°$$
$$V_s = I_s\,[0.05 + j0.5] + 250\,[\cos 0° + j\sin 0°]$$

Load 1: (lagging = inductive)

$pf = 0.8 = \theta_1$ $P_1 = 8$ kW

$|S_1| = \dfrac{P_1}{pf} = 10000$ VA



$Q_1 = |S_1|\sin\theta = 6000$ VAR

$S_1 = P_1 + jQ_1 = 8000 + j6000$

$V_1 \cdot I_1^* = S_1$

$250\ I_1^* = 8000 + j6000 \implies I_1^* = 32 + j24$ (rms)

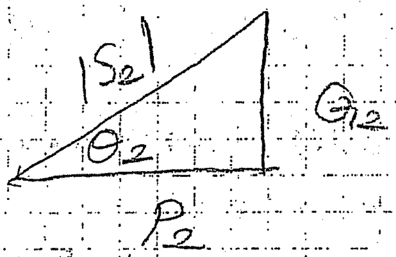
$\boxed{I_1 = 32 - j24 \text{ (rms)}}$

Load 2: (leading ⟹ capacitive)

$|S_2| = 20000\ VA$

$Q_2 < 0$

$P_2 = |S_2| \cdot pf$



$P_2 = 20000 * 0.6 = 12000\ Watts$

$Q_2 = -\sqrt{S_2^2 - P_2^2} = -16000\ VAR$

since capacitive

$S_2 = P_2 + jQ_2$

$= 12000 - j16000\ VA$

$V_2 I_2^* = S_2$

$250\angle 0^\circ\ I_2^* = 12000 - j16000$

$I_2^* = 48 - 64j$

$I_2 = 48 + 64j$ (rms)

Load 3: $Z = 2.5 + j5.0\ \Omega$

$I_3 = \frac{V_3}{Z} = \frac{250\angle 0^\circ}{2.5 + j5.0\ \Omega} = \frac{250(\cos 0^\circ + j\sin 0^\circ)}{2.5 + j5.0}$

$I_3 = \frac{100}{1+2j} = \frac{100 - 200j}{5} = 20 - 40j$

$(1-2j)$

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$$I_s = I_1 + I_2 + I_3$$

$$I_s = (32 - 24j) + (48 + 64j) + (20 - 40j)$$

$$I_s = 100 + 0j$$

$$V_s = 250\angle 0^\circ + I_s\,(0.05 + j\,0.5)$$

$$V_s = 250(\cos 0^\circ + j\sin 0^\circ) + 100\,(0.05 + j\,0.5)$$

$$V_s = 250 + 5 + 50j$$

$$V_s = 255 + 50j \quad (\text{rms}) \longrightarrow \text{phasor}$$

$$|V_s| = \sqrt{255^2 + 50^2} = 259.86$$

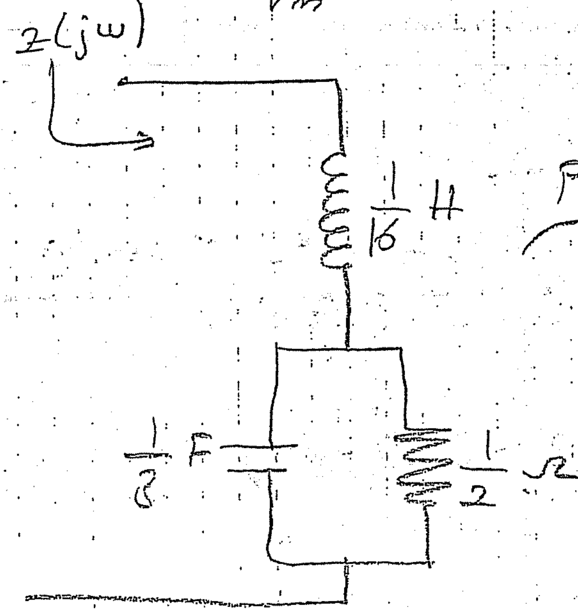
$$\angle V_s = \arctan\left|\frac{50}{255}\right| = 11.09^\circ$$

$$V_s(t) = 259.86\,\cos(2\pi f t + 11.09^\circ)$$

$$V_s(t) = 259.86\,\cos(120\pi t + 11.09^\circ) \quad (\text{rms})$$

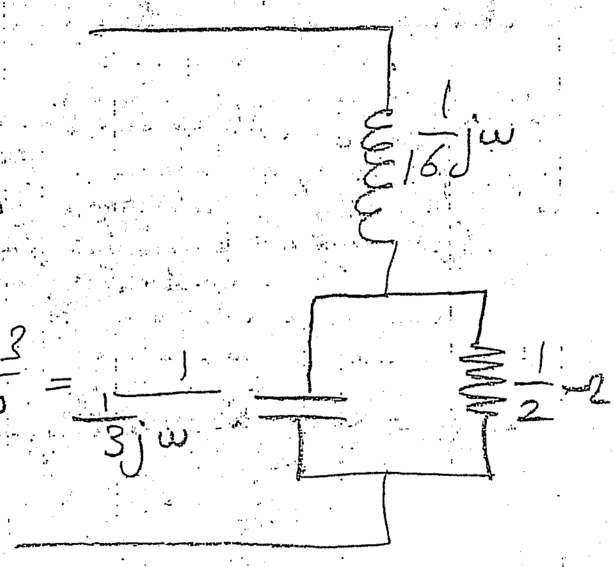
$$V_s(t) = \underbrace{\sqrt{2}\,(259.86)}_{V_m}\,\cos(120\pi t + 11.09^\circ)$$

Ex:



phasor

$$\frac{3}{j\omega} = \frac{1}{\frac{1}{3}j\omega}$$



$$z(j\omega) = \frac{j\omega}{16} + \frac{\frac{3}{j\omega} * \frac{1}{2}}{\frac{3}{j\omega} + \frac{1}{2}} = \frac{j\omega}{16} + \frac{3}{6+j\omega}$$

$$(6+j\omega) \quad (16)$$

$$z(j\omega) = \frac{6j\omega - \omega^2 + 48}{16(6+j\omega)} = \frac{36j\omega + \cancel{6\omega^2} - \cancel{6\omega^2} + j\omega^3 + 288 - 48j\omega}{16(36+\omega^2)}$$

$$(6 - j\omega)$$

$$z(j\omega) = \frac{j\omega^3 - 12j\omega + 288}{16(36+\omega^2)} = \frac{288 + j\omega(\omega^2 - 12)}{16(36+\omega^2)}$$

$\omega = 0$ or $\omega = \sqrt{12}$ rad/sec.

$\omega = 0 \Rightarrow z(j\omega) = \frac{288}{16*36} \longrightarrow$ real number

$z(j\omega) \longrightarrow$ purely resistive

$\omega = \sqrt{12} \Rightarrow z(j\omega) = \frac{288}{16(36+12)} = \frac{288}{16*48} = \frac{6}{16} \Rightarrow$ purely resistive

$0 < \omega < \sqrt{12}$ $\quad z(j\omega) = A - Bj \quad B > 0$

$\Rightarrow$ capacitive

$\sqrt{12} < \omega$ $\quad z(j\omega) = A + jC \quad C > 0$

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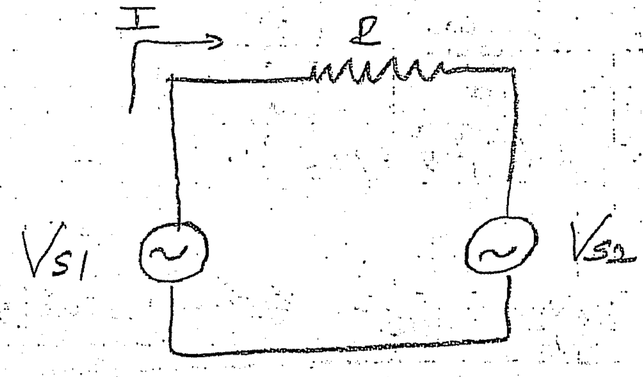
Superposition in Power Calculations:



$I_1 \rightarrow V_{S1} = V_{m1} \cos(\omega_1 t + \phi_1)$

$I_2 \rightarrow V_{S2} = V_{m2} \cos(\omega_2 t + \phi_2)$

$I(t) = I_1(t) - I_2(t)$

Instantaneous Power over Resistor R:

$$p(t) = R \cdot I^2(t)$$

$$p(t) = R\,(I_1(t) - I_2(t))^2$$

$$= \underbrace{R \cdot I_1^2(t)}_{} + \underbrace{R\, I_2^2(t)}_{} - \overbrace{2 R\, I_1(t)\, I_2(t)}^{\neq 0}$$

$$= P_1(t) + P_2(t)$$

$P(t) \neq P_1(t) + P_2(t)$

* Superposition does not apply for instantaneous power
$P(t) \neq P_1(t) + P_2(t)$

* Now consider average power

$$P_{avg} = P = \frac{1}{T} \int_0^T p(t)\, dt$$

$$P = \frac{1}{T} \int_0^T \left[R\, I_1^2(t) + R\, I_2^2(t) - 2R\, I_1(t)\, I_2(t) \right] dt$$

$$P = \frac{1}{T} \int_0^T \left[P_1(t) + P_2(t) - 2R\, I_1(t)\, I_2(t) \right] dt$$

$$P = P_{1avg} + P_{2avg} - \frac{2R}{T}\int_0^T I_1(t)\, I_2(t)\, dt$$

$$\int_0^T I_1(t)\, I_2(t) = I_{m1} I_{m2} \int_0^T \cos(\omega_1 t + \phi_1)\cos(\omega_2 t + \phi_2)\, dt$$

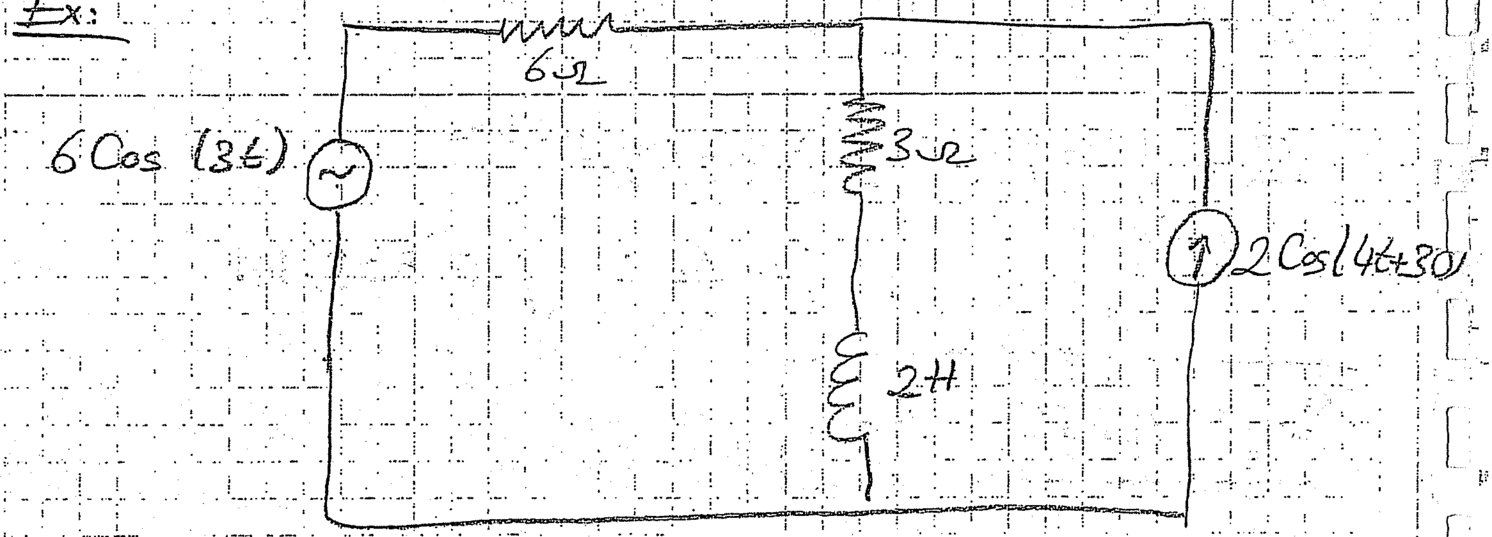
$$= \begin{cases} \frac{1}{2} I_{m1} I_{m2}\, T \cos(\phi_1 - \phi_2) \; ; & \omega_1 = \omega_2 \\ 0 \; ; & \omega_1 \neq \omega_2 \end{cases}$$

* If $\omega_1 = \omega_2$, superposition does not apply for average power $P \neq P_{1avg} + P_{2avg}$

* If $\omega_1 \neq \omega_2$; superposition does apply for average power

$$P = P_{1avg} + P_{2avg}$$

Ex:



Find average power over 3Ω resistor in SSS

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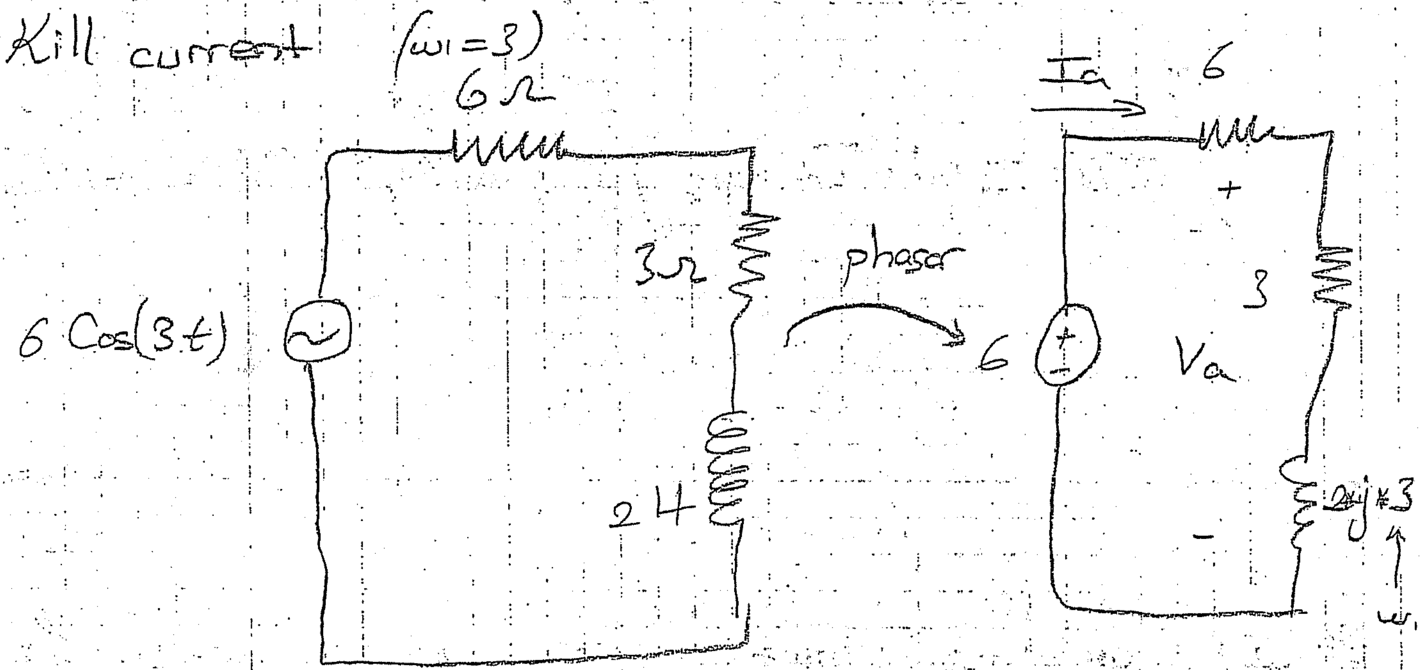
To find average power we can use superposition for individual average power due to two different sources having two different frequencies.

Kill current $(\omega_1 = 3)$



$$I_a = \frac{6}{6+3+6j} = 0.555\, e^{-j\,33.69^\circ}$$

$$V_a = I_a\,(3+6j)$$

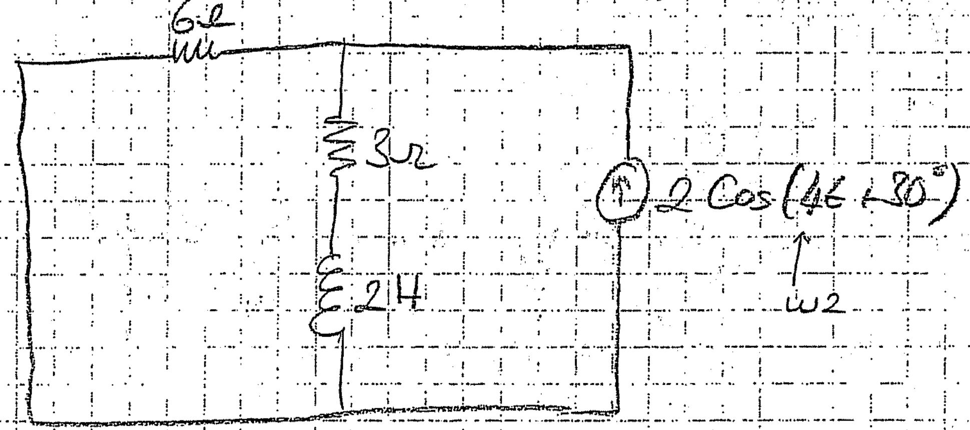
$$V_a = 3.72\, e^{j\,29.7} \text{ Volts}$$

$$S_a = \frac{1}{2} V_a I_a^* = 1.033\, e^{j\,63.43^\circ}$$

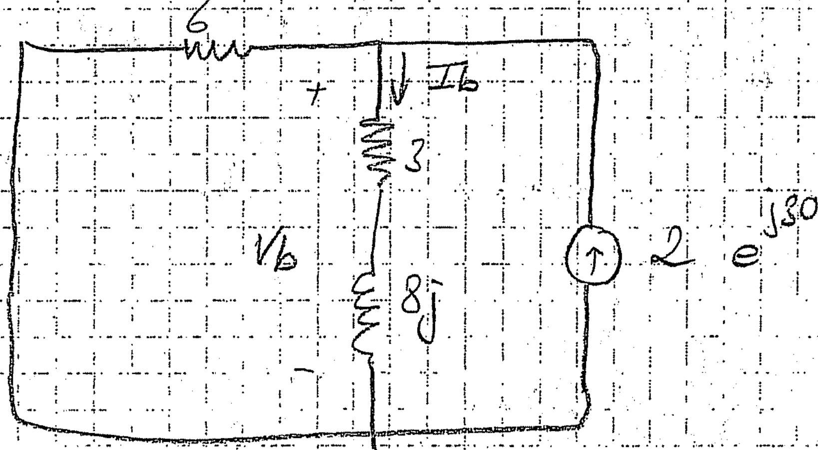
$$= (0.462 + j\,0.923)\text{ VA}$$

$$\boxed{P_{1avg} = 0.462 \text{ Watts}}$$

Kill voltage source



phasor



$$I_b = \frac{6}{6+8j+3}\, 2e^{j30^\circ} = 0.997\, e^{-j11.6^\circ}$$

$$V_b = (3+8j)\, I_b = 8.518\, e^{j57.81^\circ}$$

$$S_b = \frac{1}{2} V_b I_b^* = 4.246\, e^{j69.44^\circ}$$

$$S_b = 1.491 + j3.976 \text{ VA}$$

$$P_{2avg} = 1.491 \text{ Watts}$$

$P_{avg} = P = P_{1avg} + P_{2avg} = 1.953$ Watts
(since source frequencies are different)

Laplace Transform:

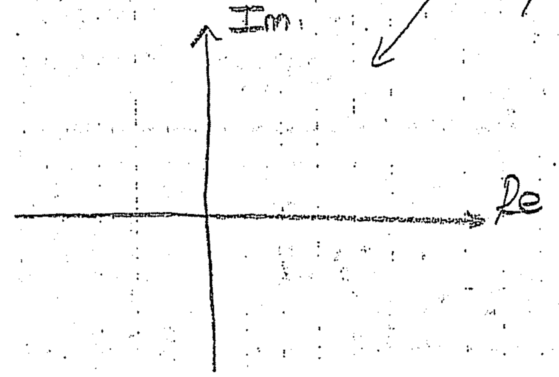
* time doman → frequency domain transformation
* diff equation → linear equation
* valid for LTI circuits.
* initial conditions are all considered (different from phasor domain)
* It gives the full-response for the circuit

Definition:

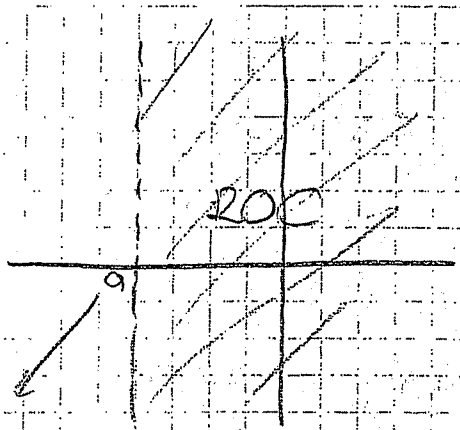
$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

s → complex variable

S-plane



ROC → (Region of convergence): The region in s-plane such that the laplace transform of the function $f(t)$ converges



ROC : $Re\{s\} > a$

will be determined by $f(t)$

Ex: $f(t) = u(t)$ find $F(s)$

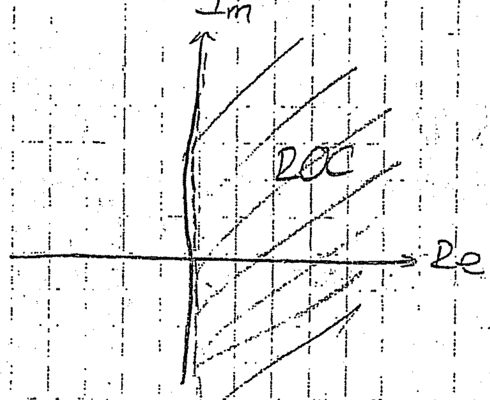
$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$F(s) = \int_{0^-}^{\infty} \underbrace{u(t)}_{1} e^{-st} dt = \int_{0^-}^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{0^-}^{\infty}$$

$$= -\frac{1}{s}\left[e^{-\infty s} - e^{-0 s}\right] = -\frac{1}{s}\left[0 - 1\right]$$

$$= \frac{1}{s}$$

ROC: $Re\{s\} > 0$



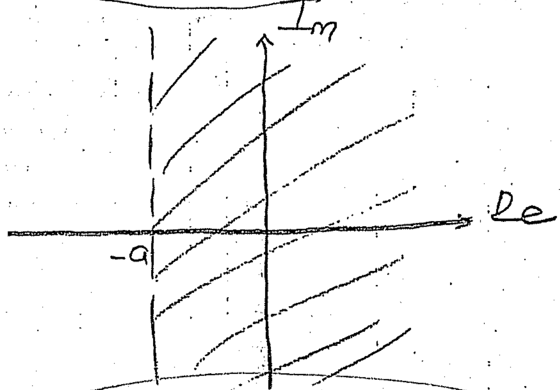
$\mathcal{L}$

Ex: $f(t) = e^{-at}u(t)$

$F(s) = \frac{1}{s+a}$ ROC: $Re\{s+a\} > 0$

$Re\{s\} + a > 0$

$Re\{s\} > -a$



Ex: $f(t) = \delta(t)$ $f(t)\delta(t) = f(0)\delta(t)$

$$F(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{\infty} \delta(t) \underbrace{e^{-s*0}}_{1} dt = \int_{0^-}^{\infty} \delta(t) dt = 1$$

$\delta(t) \xrightarrow{\mathcal{L}} 1$ ROC: whole s-plane

Inverse Laplace Transform:

$\mathcal{L}^{-1}[F(s)] = f(t)$ $$f(t) = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s) e^{st} ds$$

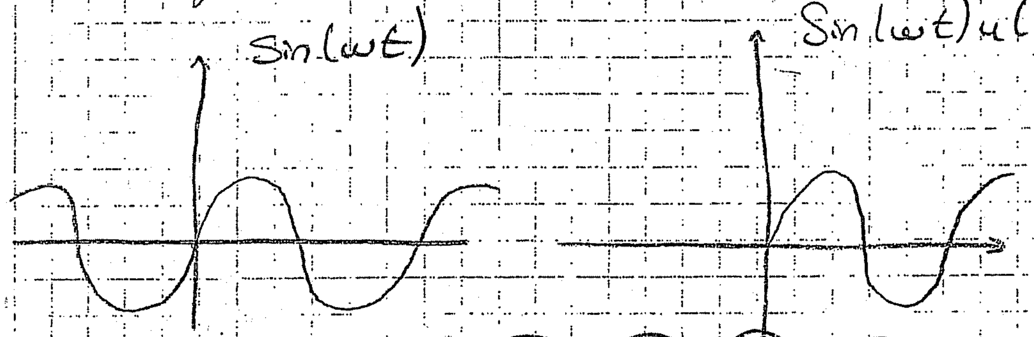
uniqueness property:

$f(t) \xrightarrow{\mathcal{L}} F(s) \Longleftrightarrow \mathcal{L}^{-1}[F(s)] = f(t)$

linearity:

$$\mathcal{L}\left[A f_1(t) + B f_2(t)\right] = A F_1(s) + B F_2(s)$$

Ex: $f(t) = A \sin(\omega t) u(t)$



$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \qquad \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$f(t) = A\sin(\omega t)u(t) = \frac{e^{j\omega t} - e^{j\omega t}}{2j}$$

$$F(s) = \int_{0^-}^{\infty} \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right] u(t)\, e^{-st}\, dt$$

$$= \frac{1}{2j} \int_{0^-}^{\infty} \left[e^{(j\omega - s)t} - e^{-(j\omega + s)t}\right] dt$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{1}{2j}\left[\frac{1}{\underset{(s+j\omega)}{s-j\omega}} - \frac{1}{\underset{(s-j\omega)}{s+j\omega}}\right]$$

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$$F(s) = \frac{1}{2j}\left[\frac{s+j\omega-(s-j\omega)}{+\omega^2+s^2}\right] = \frac{1}{\cancel{2j}}\,\frac{\cancel{2j}\omega}{(s^2+\omega^2)} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin(\omega t)\} \longrightarrow \frac{\omega}{s^2+\omega^2}$$

<u>Integration:</u>

$$k(t) = \int_{0^-}^{t} f(\tau)\,d\tau$$

$$\mathcal{L}\{k(t)\} = \mathcal{L}\left\{\int_{0^-}^{t} f(\tau)\,d\tau\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(\tau)\,d\tau\right]e^{-st}\,dt$$

(use integration by parts)

$$y = \int_{0^-}^{t} f(\tau)\,d\tau \qquad dx = e^{-st}\,dt$$

$$dy = f(t) \qquad x = -\frac{1}{s}e^{-st}$$

$$\int_a^b y\,dx = yx\Big]_a^b - \int_a^b x\,dy = \left[\int_{0^-}^{t} f(\tau)\,d\tau\right]\left(-\frac{1}{s}e^{-st}\right)\Bigg|_0^\infty$$

$$-\int_{0^-}^{\infty} f(t)\left(-\frac{1}{s}e^{-st}\right)dt$$

$$= \Bigg[\underbrace{\left(\int_{0^-}^{\infty} f(z)\,dz\right)}_{M} \underbrace{\left(-\frac{1}{s} e^{-s\infty}\right)}_{0} - \underbrace{\left(\int_{0^-}^{0^-} f(z)\,dz\right)}_{0} \underbrace{\left(-\frac{1}{s} e^{-0*s}\right)}_{-1/s} \Bigg]$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{0}$$

$$+\frac{1}{s}\overbrace{\int_{0^-}^{\infty} f(t)\,e^{-st}\,dt}^{F(s)/s} = \frac{F(s)}{s}$$

$$\mathcal{L}\left\{\int_{0^-}^{t} k(t)\,dt\right\} = \frac{K(s)}{s}$$

Ex: $f(t) = t\,u(t) = r(t)$

$$r(t) = \int_{0^-}^{t} u(z)\,dz$$

$$\mathcal{L}\{r(t)\} = \mathcal{L}\left\{\int_{0^-}^{t} u(z)\,dz\right\} = \frac{1}{s}\mathcal{L}\{u(t)\} = \frac{1}{s} * \frac{1}{s}$$

$$= \frac{1}{s^2}$$

Differentiation:

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{d^2}{dt^2} f(t)\right\} = s^2F(s) - sf(0^-) - f'(0^-)$$ HW proof

$$\mathcal{L}\left\{\frac{d^n}{dt^n} f(t)\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - s^{n-3}f''(0^-) - \cdots - sf^{n-2}(0^-) - f^{n-1}(0^-)$$

$$\mathcal{L}\left\{\sin(\omega t)\right\} = \frac{\omega}{s^2+\omega^2}$$

$$\cos(\omega t) = \frac{1}{\omega}\frac{d}{dt}\sin(\omega t)$$

$$\mathcal{L}\left\{\cos(\omega t)\right\} = \frac{1}{\omega}\mathcal{L}\left\{\frac{d}{dt}\sin(\omega t)\right\}$$

$$= \frac{1}{\omega}\left[s\,\mathcal{L}\left\{\sin(\omega t)\right\} - \underbrace{\sin(\omega 0^-)}_{0}\right]$$

$$= \frac{1}{\omega}\left[s\,\frac{\omega}{s^2+\omega^2} - 0\right] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\left\{\cos(\omega t)\right\} = \frac{s}{s^2+\omega^2}$$

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Differentiation in s-domain:

$$-t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}$$

Shift in time domain:

$$\mathcal{L}\{f(t-a)\} = e^{-sa}F(s)$$

Shift in frequency domain:

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

$$\mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s+a)^2+\omega^2}$$

Scaling Property:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) \quad a>0$$

Pole - Zero Diagram:

$$F(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s - b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s - a_0}$$

$$F(s) = \left(\frac{b_m}{a_n}\right)\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

scalar

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$p_1, p_2, ---- p_n \longrightarrow$ poles $\longrightarrow$ X

$z_1, z_2, ----, z_n \longrightarrow$ zeros $\longrightarrow$ O

Ex: $f(t) = \left[e^{-2t} + \cos(2t) - \sin(2t) \right] u(t)$

find pole zero diagram

$F(s) = ?$

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{s+2} + \frac{s}{s^2+4} - \frac{2}{s^2+4}$$

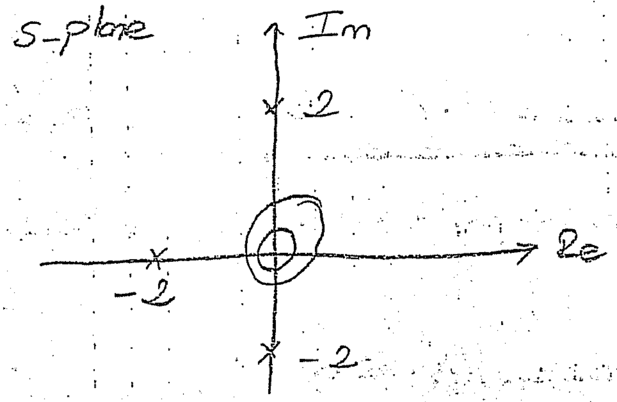
$(s^2+4) \quad (s+2) \quad (s+2)$

$$F(s) = \frac{s^2+\cancel{4}+s^2+\cancel{2s}-\cancel{2s}-\cancel{4}}{(s^2+4)(s+2)} = \frac{2s^2}{(s+2)(s+2j)(s-2j)}$$

$p_1 = -2 \quad p_2 = -2j \quad p_3 = 2j$

$z_1 = 0 \quad z_2 = 0$

Pole - zero diagram:



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Note that: The poles (p_i values) give the natural frequencies of the circuit when there is no input.

Relation between the zeros and poles in s_plane:

Ex: $\alpha > 0$ $\quad f(t) = e^{-\alpha t} \cos(\omega_0 t) u(t)$

$$F(s) = \frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2} = \frac{N(s)}{D(s)}$$

poles $\Rightarrow$ $D(s) = 0$ (roots)

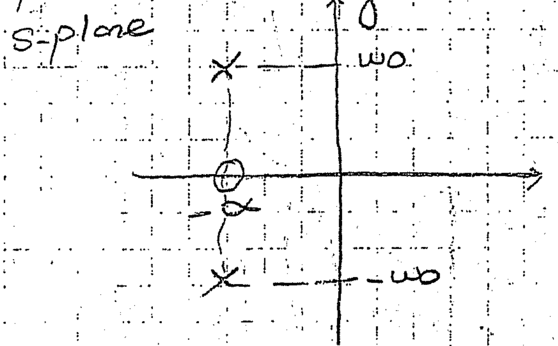
zeros $\Rightarrow$ $N(s) = 0$ (roots)

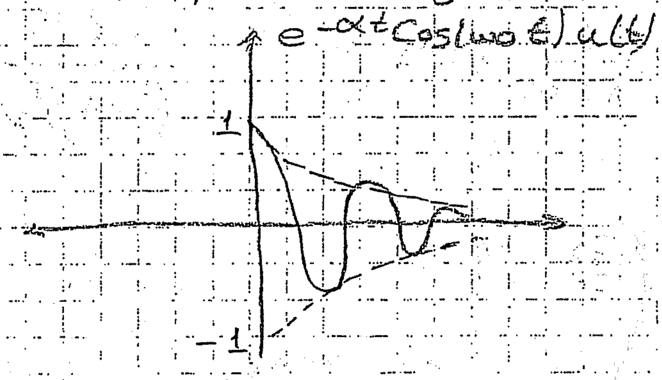
$z_1 = -\alpha$

$p_1 = -\alpha + j\omega_0$

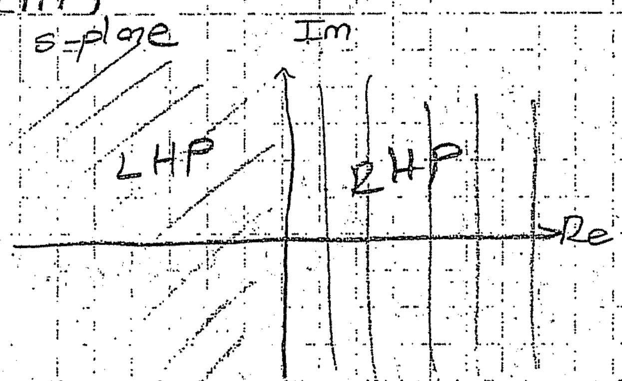
$p_2 = -\alpha - j\omega_0$

pole - zero diagram:





* If the poles of a circuit are on the left half plane (LHP)



the time waveform is bounded otherwise unbounded

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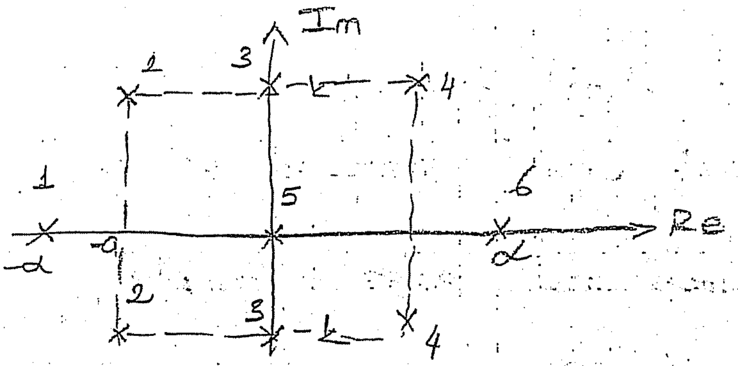
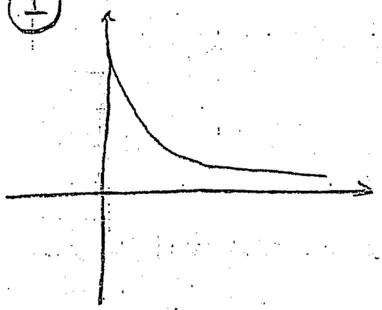


1



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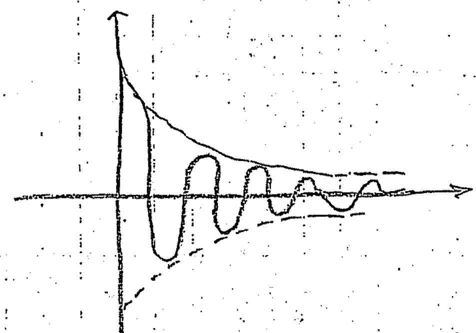
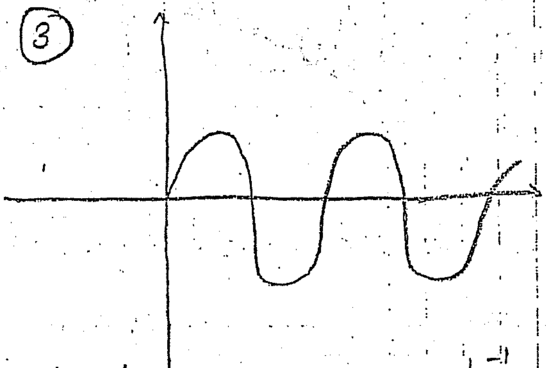


$$\frac{1}{s+\alpha} \xrightarrow{\mathcal{L}^{-1}} e^{-\alpha t}$$

$$\frac{A_1^*}{(s+a+jk)} + \frac{A_1}{(s+a-jk)} \xrightarrow{\mathcal{L}^{-1}} e^{-at}\left[K_1\cos(kt)+K_2\sin(kt)\right]$$

3



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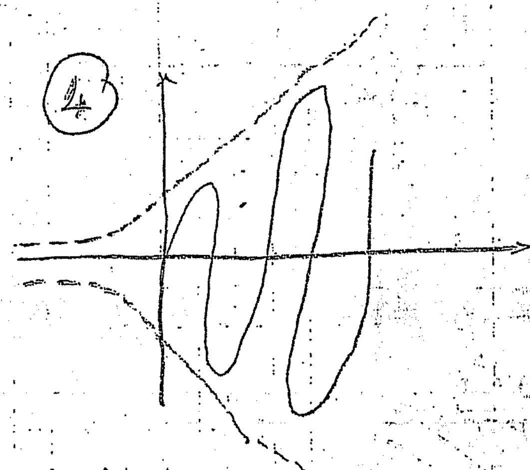
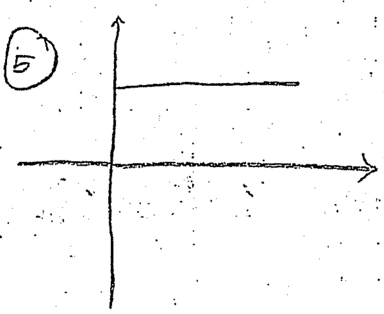


$$\frac{A_1^*}{s+jk} + \frac{A_1}{s-jk} \xrightarrow{\mathcal{L}^{-1}} K_1\cos(kt)+K_2\sin(kt)$$

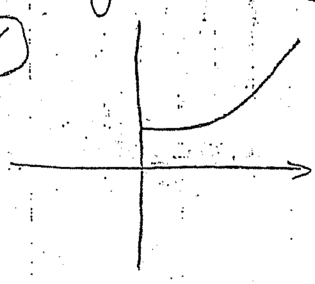
4 $$\frac{A_1}{-s+a+jk} + \frac{A_1^*}{s-a-jk} \xrightarrow{\mathcal{L}^{-1}} e^{at}\left[K_1\cos(kt)+K_2\sin(kt)\right]$$

5

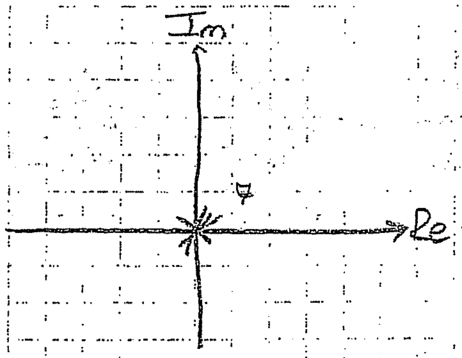


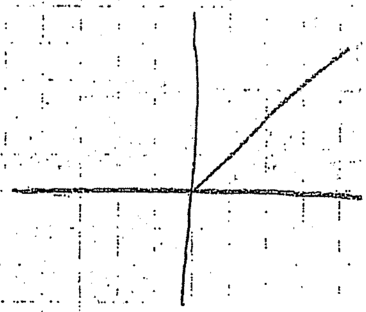
$$\frac{1}{s} \xrightarrow{\mathcal{L}^{-1}} u(t)$$

6

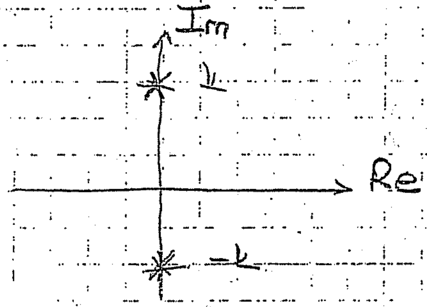


$$\frac{1}{s-\alpha} \xrightarrow{\mathcal{L}^{-1}} e^{\alpha t}u(t)$$

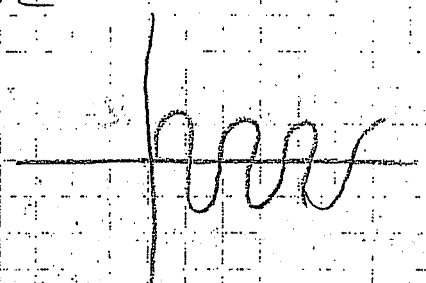


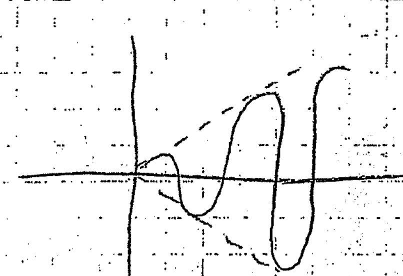


$$\frac{1}{s^2} \xrightarrow{\mathcal{L}^{-1}} r(t)$$



$$\frac{1}{(s^2+\omega^2)^2} \xrightarrow{\mathcal{L}^{-1}} \left[A_1 \cos(\omega t) + A_2 \sin(\omega t)\right] + t\left[A_3 \cos(\omega t) + A_4 \sin(\omega t)\right]$$





Find inverse laplace transform:

$$\frac{A(s)}{B(s)} = \underbrace{\rho(s)}_{\text{improper expression}} + \underbrace{\frac{N(s)}{D(s)}}_{\text{proper rational function}} \qquad \deg(N(s)) < \deg(D(s))$$

Ex: (For proper part)

$$F(s) = \frac{K(s-z_1)}{(s-p_1)(s-p_2)(s-p_3)} \underset{\text{partial fraction expansion}}{=} \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3}$$

Find $A_1; A_2; A_3$

$A_1; A_2; A_3 \rightarrow$ residues

$$A_1 = (s-p_1)\,F(s)\Big|_{s=p_1} = \frac{K\,(s-z_1)}{(s-p_1)\,(s-p_3)}\Bigg|_{s=p_1}$$

Ex: $F(s) = \dfrac{2(s+3)}{s(s+1)(s+2)} = \dfrac{K_1}{s} + \dfrac{K_2}{s+1} + \dfrac{K_3}{s+2}$

$$(s+1)(s+2) \quad s(s+2) \quad s(s+1)$$

$$\Longrightarrow \quad 2(s+3) = K_1(s+1)(s+2) + K_2 s(s+2) + K_3 s(s+1)$$

$$2s+6 = s^2K_1 + 3sK_1 + 2K_1 + K_2s^2 + 2sK_2 + K_3s^2 + K_3s$$

$K_1 + K_2 + K_3 = 0 \Rightarrow -/\;K_2 + K_3 = -3$

$3K_1 + 2K_2 + K_3 = 2 \qquad 2K_2 + K_3 = -7$

$2K_1 = 6 \Rightarrow K_1 = 3 \qquad K_2 = -4 \quad K_3 = 1$

$$F(s) = \frac{3}{s} + \frac{-4}{s+1} + \frac{1}{s+2}$$

$\mathcal{L}^{-1}\big\downarrow$

$$f(t) = \left[3 - 4e^{-t} + e^{-2t}\right]u(t)$$

(Simple poles) $\rightarrow$ all poles are real and distinct from each other complex but simple poles

* If $F(s)$ is a rational expression with real coefficient then (if exist) the complex zeros and poles should appear in conjugate pairs

assume poles exist $p_1 = -\alpha + j\beta$; $p_2 = -\alpha - j\beta$

$$F(s) = \underbrace{\_\_\_\_}_{\text{other poles}} + \frac{K}{s+\alpha-j\beta} + \frac{K^*}{s-\alpha-j\beta} + \underbrace{\_\_\_\_}_{\text{other poles}}$$

$K \Longrightarrow$ complex number

$K = |K| e^{j\theta}$ $\quad K^* = |K| e^{-j\theta}$

$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t) = \underbrace{\_\_\_\_}_{\substack{\text{terms due to}\\ \text{other terms}}} + |K| e^{j\theta} e^{(-\alpha+j\beta)t} + |K| e^{-j\theta} e^{(-\alpha-j\beta)t} + \_\_$$

$$f(t) = \_\_\_\_ + |K| e^{-\alpha t} \left[\frac{e^{j(\theta+\beta t)} + e^{-j(\theta+\beta t)}}{2} \right] \times 2 + \_\_\_\_$$

$$f(t) = \left[\_\_\_\_ + 2|K| e^{-\alpha t} \cos(\beta t + \theta) + \_\_\_\_ \right] u(t)$$

Ex: $$F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)} = \frac{20(s+3)}{(s+1)\left[(s+1)^2+2^2\right]}$$

poles $p_1 = -1$ $\quad p_2 = -1+2j$ $\quad p_3 = -1-2j$

zeros $z_1 = -3$

$$F(s) = \frac{K_1}{s+1} + \frac{K_2}{s+1-2j} + \frac{K_2^*}{s+1+2j}$$

$$K_2 = (s+1-2j)\,F(s)\Big|_{s=-1+2j}$$

$$K_2 = \frac{20(s+3)}{(s+1)\,(s+1+2j)}\Bigg|_{s=-1+2j} = \frac{20(-1+2j+3)}{(-1+2j+1)\,(-1+2j+1+2j)}$$

$$= \frac{20(2+2j)}{2j\,4j} = \frac{-40(1+j)}{8}$$

$$K_2 = -5-5j$$

$$|K_2| = \sqrt{(-5)^2+(-5)^2} = 5\sqrt{2}$$

$$K_2 = 5\sqrt{2}\,e^{j\frac{5\pi}{4}} \Rightarrow K_2^* = 5\sqrt{2}\,e^{-j\frac{5\pi}{4}}$$

__H.W.__ $K_1 = 10$

$$f(t) = \left[10\,e^{-t} + 2\,|5\sqrt{2}|\,e^{-t}\cos\left(2t+\frac{5\pi}{4}\right)\right]u(t)$$

<u>Multiple pole</u>

$$F(s) = \frac{N(s)}{(s-p_1)^n} = \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_1)^2} + \cdots + \frac{A_n}{(s-p_1)^n}$$

$$(s-p_1)^{n-1} \quad (s-p_1)^{n-2}$$

$$A_n = F(s)\,(s-p_1)^n \Big|_{s=p_1}$$

$$A_{n-1} = \frac{d}{ds}\left[F(s)\,(s-p_1)^n\right] \Big|_{s=p_1}$$

$$A_{n-2} = \frac{d^2}{ds^2}\left[F(s)\,(s-p_1)^n\right] \Big|_{s=p_1}$$

<u>Ex:</u> <u>Improper rational expression</u> $\deg(N(s)) > \deg(D(s))$

$$F(s) = \frac{s^3+6s^2+12s+8}{s^2+4s+3} = \frac{N(s)}{D(s)} = s+2 + \frac{s+2}{s^2+4s+3}$$

$s+2$ → polynomial improper

$\frac{s+2}{s^2+4s+3}$ → proper rational expression

$$\frac{N(s)}{D(s)} = \left\{ \begin{array}{l|l} s^3+6s^2+12s+8 & s^2+4s+3 \\ \cline{2-2} s^3+4s^2+3s & s+2 \\ \cline{1-1} \quad 2s^2+9s+8 & \\ \quad 2s^2+8s+6 & \\ \cline{1-1} \qquad s+2 & \end{array} \right.$$

$$\mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\left\{\frac{d}{dt}\delta(t)\right\} = s\,\mathcal{L}\{\delta(t)\} - \delta(0^-)$$

$$= s \cdot 1 - 0 = s$$

$$F(s) = s + 2 + \frac{1/2}{s+1} + \frac{1/2}{s+3}$$

$\mathcal{L}^{-1}$

$$f(t) = \dot{\delta}(t) + 2\delta(t) + \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Initial and Final Value Theorems:

Final value $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

Initial value theorem $\lim_{t \to 0^+} f(t) = \lim_{s \to 0} sF(s)$

Constraints

1. Initial value theorem is valid when $F(s)$ is a proper function.

$$F(s) = \frac{s^2+s+1}{s^2+s-1} = \frac{N(s)}{D(s)}$$

$\deg(N(s)) = 2$

$\deg(D(s)) = 2$

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$$\begin{array}{r|l} s^2+s+1 & s^2+s-1 \\ -\;s^2+s-1 & 1 \\ \hline 2 & \end{array}$$

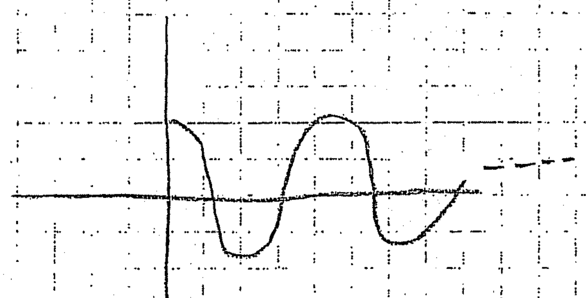
$$F(s) = \frac{N(s)}{D(s)} = 1 + \frac{2}{s^2+s-1} \longrightarrow \text{improper}$$

real number (polynomial)

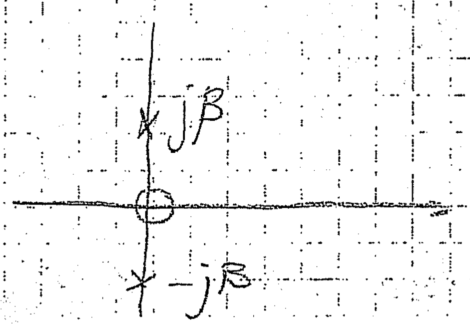
* In order to obtain a proper function $\deg(N(s)) < \deg(D(s))$

(2) Final value theorem is valid when the poles of $sF(s)$ are all in left half plane

<u>Ex:</u> $\lim_{t \to \infty} \cos(\beta t) = $ (unknown)



$$\cos(\beta t) \longrightarrow \frac{s}{s^2+\beta^2}$$



poles are at imaginary axis (these poles are not in left half plane)

(unknown)

$$\lim_{t \to \infty} \cos(\beta t) \overset{?}{=} \lim_{s \to 0} s \cdot \frac{s}{s^2+\beta^2} = 0$$

contradiction

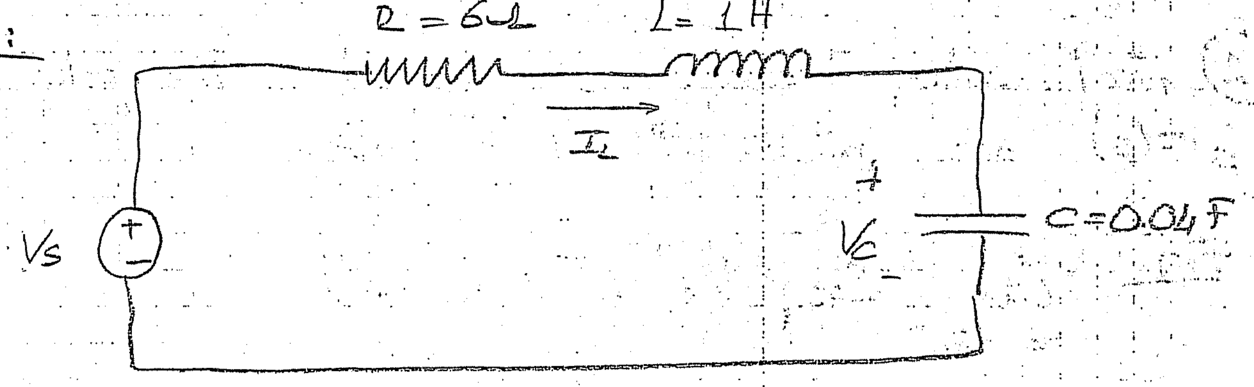
Circuit Analysis in Laplace Domain:

① Find differential equation

② Transform the differential equation to Laplace domain.

③ Find the answer and again turn to time domain using $\mathcal{L}^{-1}$

Ex:



$V_c(0^-) = 1V$

$I_L(0^-) = 5\ A$

$V_s(t) = 12 \sin(5t) \quad t \geq 0$

KVL $\quad V_s = V_R + V_L + V_C$

$$V_s = I_L \cdot R + L \frac{dI_L}{dt} + V_C$$

$I_C = C \dfrac{dV_C}{dt}$

$$\int_{0^-}^{t} \frac{1}{C} I_C(t')dt' \qquad \int_{V_C(0^-)}^{V_C(t)} dV_C \qquad \frac{1}{C}\int_{0^-}^{t} I_C(t')dt' = V_C(t) - V_C(0^-)$$

$$V_c(t) = V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} I_L(t')\,dt'$$

$$V_s = I_L \cdot R + L\frac{dI_L}{dt} + V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} I_L(t')\,dt'$$

(integro-differential equation)

$\mathcal{L}$

$$V_s(s) = R\,I_L(s) + L\left[s\,I_L(s) - I_L(0^-)\right] + \frac{V_c(0^-)}{s}$$

$$+\frac{1}{C}\cdot\frac{I_L}{s}$$

integralin Laplace'ı nasıl oluyor.

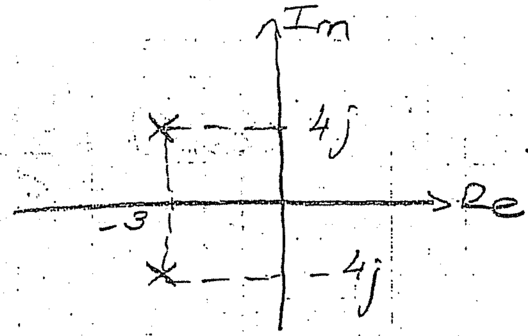
$$I_L(s) = \frac{s\,V_s(s)\,\frac{1}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} + \frac{s\,I_L(0^-) - \frac{1}{L}V_c(0^-)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\underbrace{I_L(s)}_{\text{complete response}} = \underbrace{\frac{s\,V_s(s)}{(s+3)^2+4^2}}_{\substack{\text{zero-state response}\\ I_{zs}(s)}} + \underbrace{\frac{5s-1}{(s+3)^2+4^2}}_{\substack{\text{zero-input response}\\ I_{zi}(s)}}$$

pole location

$s_1 = -3+4j$

$s_2 = -3-4j$



Let's find zero-state response $I_{zs}(t)$

$$V_s(s) = 12 * \frac{5}{s^2+5^2} = \frac{60}{s^2+5^2}$$

$$I_{zs}(s) = \frac{s \cdot 60}{\underbrace{(s+3)}_{s}{}^2+5^2} * \frac{1}{(s+3)^2+4^2} = \frac{K_1}{s+3-4j} + \frac{K_1^*}{s+3+4j} + \frac{K_2}{s-5j} + \frac{K_2^*}{s+5j}$$

$$I_{zs}(t) = 2.5\, e^{-3t} \cos(4t+90^\circ) + 2\cos(5t-90^\circ)$$

$$I_{zs}(t) = -2.5\, e^{-3t} \sin(4t) + 2\sin(5t) \quad t \geq 0$$

$$K_1 = (s+3-4j)\, I_{zs}(s)\Big|_{s=-3+4j} \qquad K_2 = (s-5j)\, I_{zs}(s)\Big|_{s=5j}$$

$$\frac{K}{(s+\alpha-j\beta)} + \frac{K^*}{(s+\alpha+j\beta)} \quad , \quad K = |K| e^{j\theta}$$

$$\mathcal{L}^{-1} \downarrow$$

$$2\,|K|\, e^{-\alpha t} \cos(\beta t + \theta)$$

$$I_{2C}(s) = \frac{5s-1}{(s+3)^2+4^2} = \frac{K_3}{s+3-4j} + \frac{K_3^*}{s+3+4j}$$

$$K_3 = \frac{5s-1}{(s+3)^2+4^2}(s+3-4j)\Bigg|_{s=-3+4j}$$

$$K_3 = \frac{5s-1}{s+3+4j}\Bigg|_{s=-3+4j} = \frac{5(-3+4j)-1}{8j}$$

$$K_3 = \frac{-16+20j}{8j} = 2j + 2.5$$

$$|K_3| = \sqrt{(+2)^2+(+2.5)^2} \cong 3.2$$

$$\angle K_3 = \theta = 38.7^\circ = \arctan\left(\frac{2}{2.5}\right)$$

$$K_3^* = 2.5 - 2j$$

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$$I_{zi}(t) = 2|K_3| e^{-3t} \cos(4t+\theta)$$

$$I_L(t) = I_{Lzs}(t) + I_{Lzi}(t)$$

$$= \underbrace{\underbrace{-2.5\, e^{-3t} \sin(4t)}_{\text{transient}} + \underbrace{2\sin(5t)}_{\text{sss}}}_{\text{zero-state response}} + \underbrace{\underbrace{6.4\, e^{-3t}(\cos(4t+\theta))}_{\text{transient}}}_{\text{zero-input response}}$$

sinusoidal steady state

Let $V_s(t) \neq 12\sin(5t)$

$$V_s(t) = \delta(t) \xrightarrow{\mathcal{L}} V_s(s) = 1$$

Find zero-state response due to this input

$$I_{Lzs}(s) = \frac{V_s * s}{(s+3)^2+4^2} = \frac{s}{(s+3)^2+4^2}$$

$$= \frac{K_1}{s+3-4j} + \frac{K_1^*}{s+3+4j}$$

$(s+3+4j)$ $\quad$ $(s+3-4j)$

$$s = K_1(s+3+4j) + K_1^*(s+3-4j)$$

$$s = (K_1 + K_1^*)s \qquad 0 = K_1(3+4j) + K_1^*(3-4j)$$

$K_1 = a + bj$

$s = (a + bj + a - bj)\,s \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2} = \frac{4}{8}$

$0 = (a+bj)(3+4j) + (a-bj)(3-4j)$

$0 = 3a - 4b + \cancel{3bj} + \cancel{4aj} + 3a - 4b - \cancel{4aj} - \cancel{3bj}$

$0 = 6a - 8b \Rightarrow b = \frac{3}{8}$

$K = \frac{4}{8} + \frac{3}{8}j$

$|K| = \frac{5}{8}$ $\quad \cancel{\angle K} = \arctan \frac{3/8}{4/8} = \theta = 36.87^\circ$

$I_{Lzs}(t) = 2|K|\, e^{-3t} \cos(4t + \theta)$

$I_{Lzs}(t) = \frac{2 * 5}{8} e^{-3t} \cos(4t + 36.87^\circ)$

zero-state response

Zero-State Response (Network function) for the previous question

$$I_{Lzs} = \frac{V_s(s)\,L \cdot s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\frac{I_{Lzs}(s)}{V_s(s)} = \frac{s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

* No initial condition (all initial conditions are zero)

* The relation between input ($V_s(s)$) and output ($I_{L_{zs}}(s)$)

Network function (Transfer function).

The relation between input and output when the initial conditions are all 0.

$$Y(s) = H(s)\,U(s)$$

output — $Y(s)$; transfer function — $H(s)$; input — $U(s)$

$$H(s) = \frac{Y(s)}{U(s)}$$

$$H(s) \xrightarrow{\mathcal{L}^{-1}} h(t)$$

$h(t)$ impulse response

$u(t) = \delta(t)$

$$u(t) = \delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t)$$

impulse response

$$h(t) \xrightarrow{\mathcal{L}} H(s)$$

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Observations:

① For LTI networks, zero-state (z.s.) response is obtained by convolution.

$$y(t) = \int_{0^-}^{\infty} h(t-\tau)\, u(\tau)\, d\tau \qquad t \geq 0$$

$y(t)$: output (zero-state)

② $Y(s) = H(s)\, U(s)$

$Y(s)$: zero-state response

Convolution Theorem:

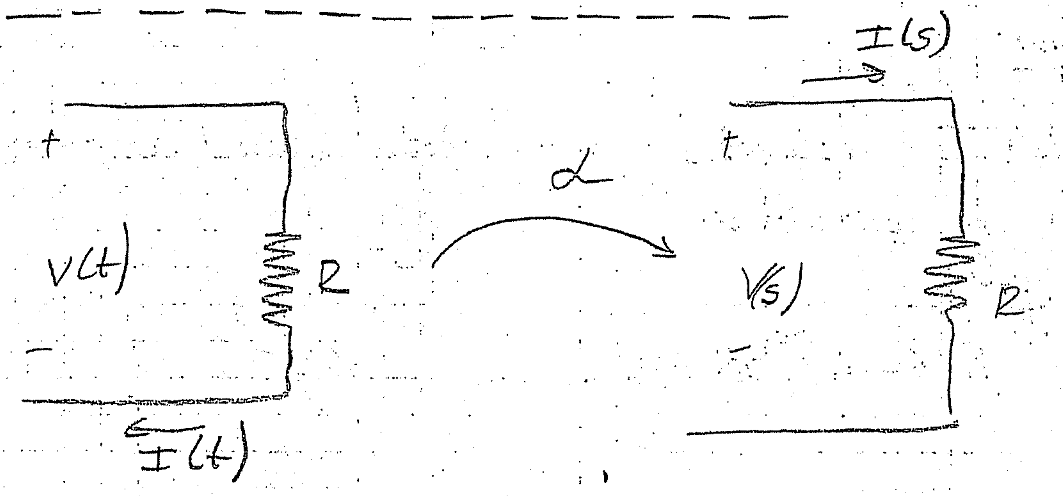
$$\int_{0^-}^{\infty} f_1(t-\tau)\, f_2(\tau)\, d\tau = \int_{0^-}^{\infty} f_2(\tau)\, f_2(t-\tau)\, d\tau = f_3(t)$$

$$\xrightarrow{\mathcal{L}} F_3(s) = F_1(s)\, F_2(s)$$

For circuit analysis in Laplace Domain:

KVL; KCL; node - mesh analysis; source conversions, dependent; independent sources

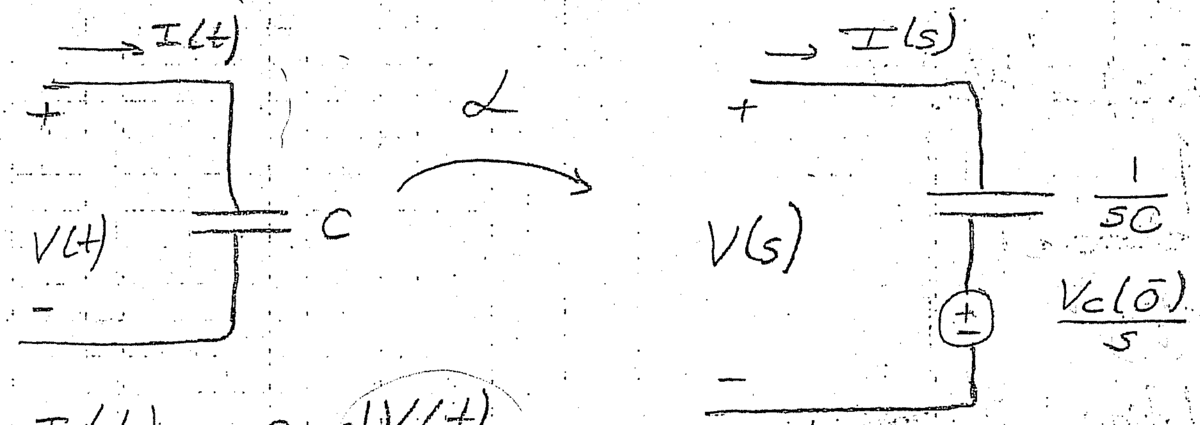
RLC elements in s-domain:



$I(t) = C\frac{dV(t)}{dt}$
$I(s) = C(sV(s) - V(0))$
$V(s) = I(s)\frac{1}{sC} + \frac{V(0)}{s}$

$V(t) = R\, I(t)$

$V(s) = R \cdot I(s)$

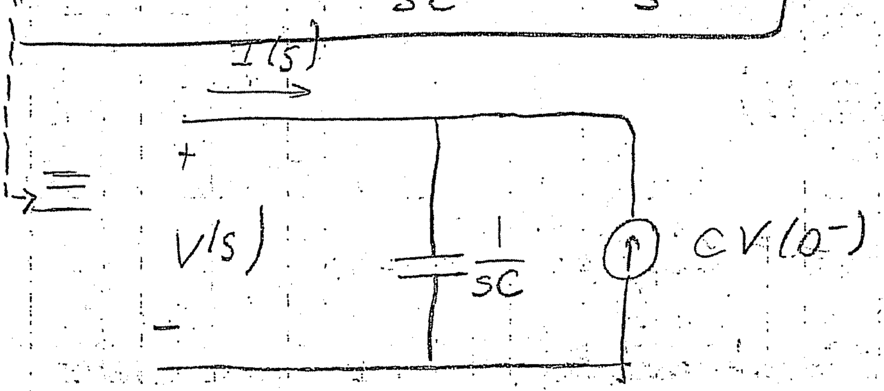


$I(t) = C\frac{dV(t)}{dt}$

$I(s) = C\left[sV(s) - V(0^-)\right]$

initial condition

$$\boxed{V(s) = I(s)\frac{1}{sC} + \frac{V(0^-)}{s}}$$



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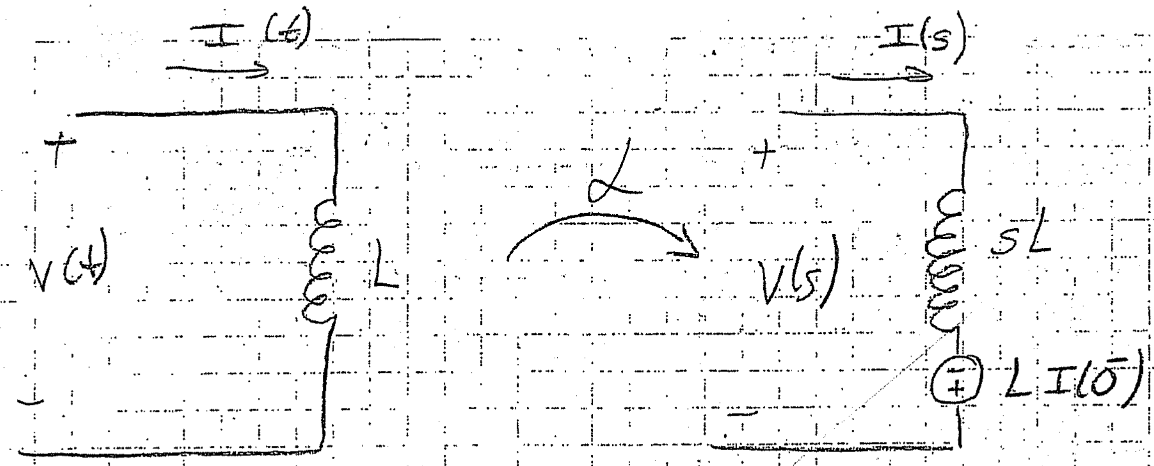
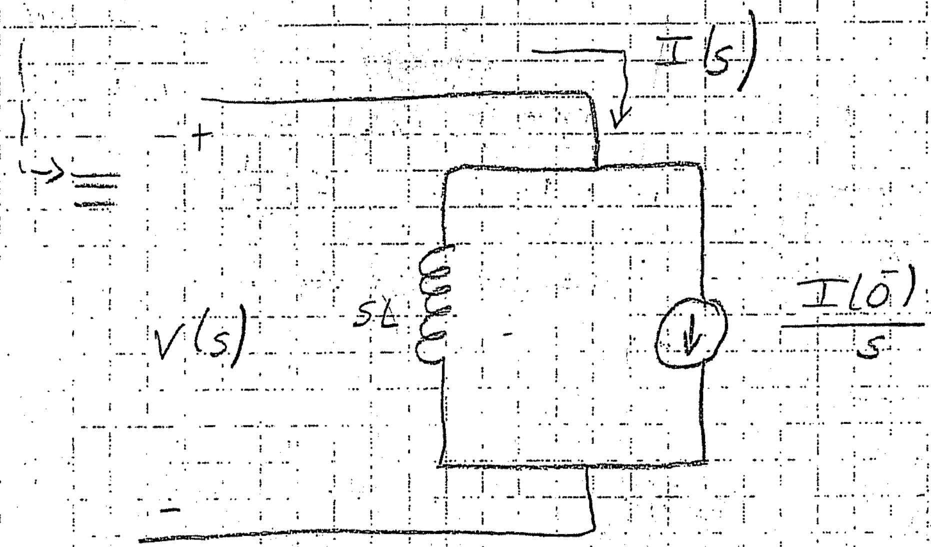
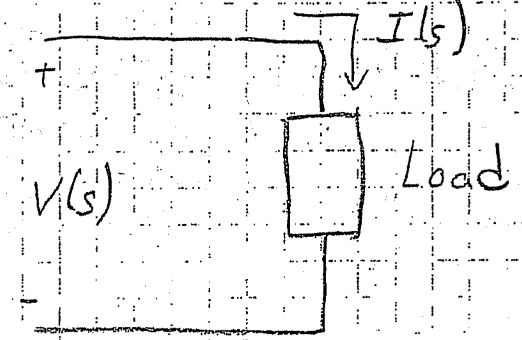


$$L\frac{dI}{dt} = V(t) \qquad L\left[sI(s) - I(0^-)\right]$$



$$I(s) = \frac{V(s)}{sL} + \frac{I(0^-)}{s}$$

Impedance and Admittance:



$$Z(s) = \left.\frac{V(s)}{I(s)}\right|$$

impedance

zero initial condition (zero-state)

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$$R \longrightarrow Z(s) = R \longrightarrow Y(s) = \frac{1}{R}$$

$$L \longrightarrow Z(s) = sL \longrightarrow Y(s) = \frac{1}{sL}$$

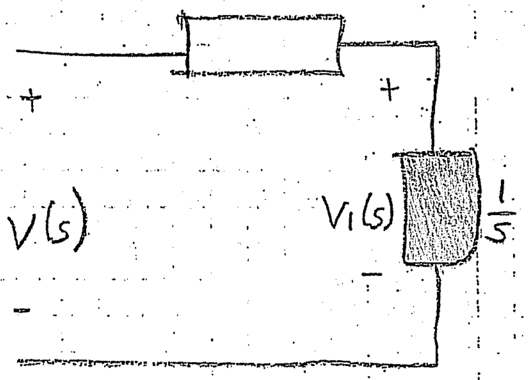
$$C \longrightarrow Z(s) = \frac{1}{sC} \longrightarrow Y(s) = sC$$

$$I_s = \frac{V(s)}{sL} + \frac{I(0)}{s}$$

$$Y(s) = \text{Admittance} = \frac{1}{Z(s)}$$

$$V_s = \frac{I_s}{C \cdot s} + \frac{V(0)}{s}$$

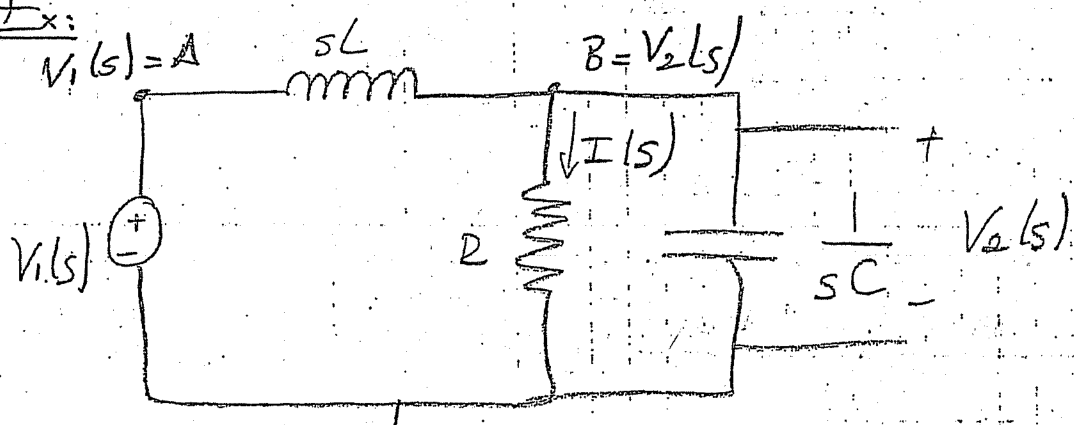
Voltage division, current division, series parallel connections
(similar to time domain resistive circuits)



$$V_1(s) = V(s)\,\frac{s}{s + \frac{1}{s}}$$

$$Z_{Total} = \frac{1}{s} + s$$

Ex:



Find $H(s) = \dfrac{V_2(s)}{V_1(s)}$

transfer function

$H_2(s) = \dfrac{I(s)}{V_1(s)}$ (another transfer function)

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$$\frac{V_1(s) - V_2(s)}{sL} = \frac{V_2(s)}{R} + \frac{V_2(s)}{\frac{1}{sC}}$$

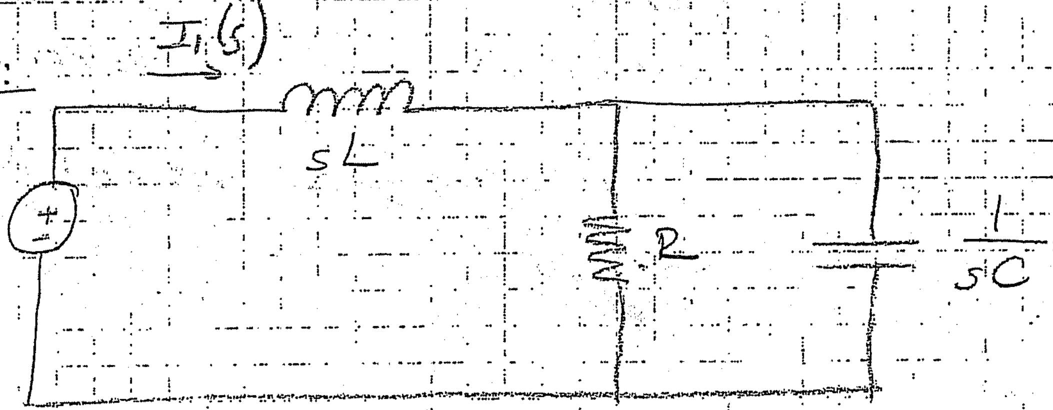
$$\frac{V_1(s)}{sL} = \frac{V_2(s)}{sL} + \frac{V_2(s)}{R} + sC\,V_2(s)$$

$$(R) \qquad (R) \qquad (sL) \qquad (RsL)$$

$$R V_1(s) = (R + sL + s^2 RLC)\, V_2(s)$$

$$\frac{V_2(s)}{V_1(s)} = H(s) = \frac{R}{R + sL + s^2 RLC} = \frac{1/LC}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Ex:



$$Z_i = \frac{V_1(s)}{I_1(s)} = sL + \left(\frac{1}{sC} \parallel R\right)$$

input impedance

$$sL + \frac{\frac{1}{sC} \cdot R}{\frac{1}{sC} + R} = Z_i(s) = sL + \frac{R}{R + sRC}$$

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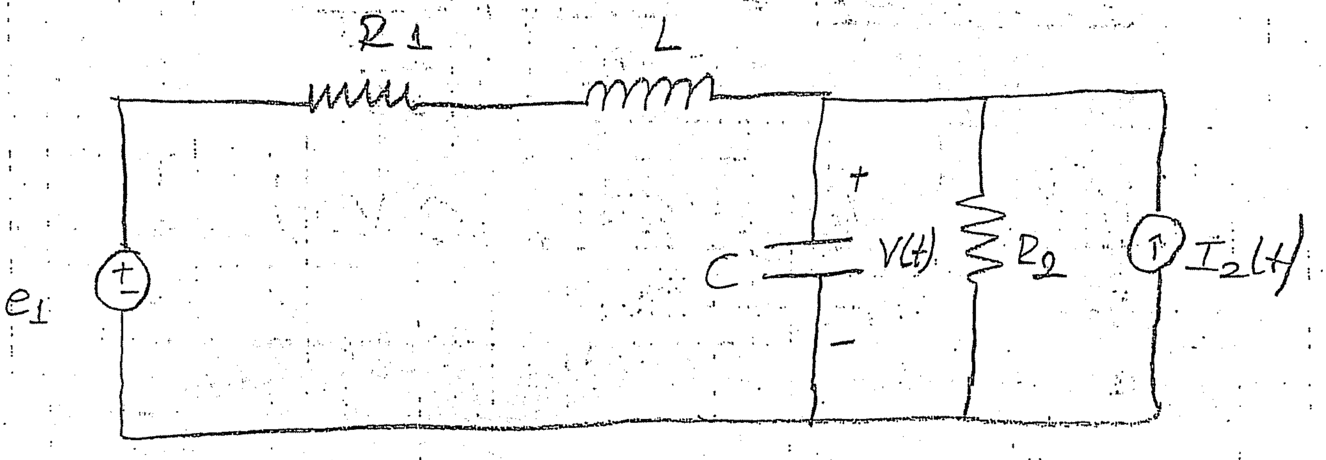
Superposition:

Ex:



$$V(s) = H_1(s)\,E_1(s) + H_2(s)\,I_2(s)$$

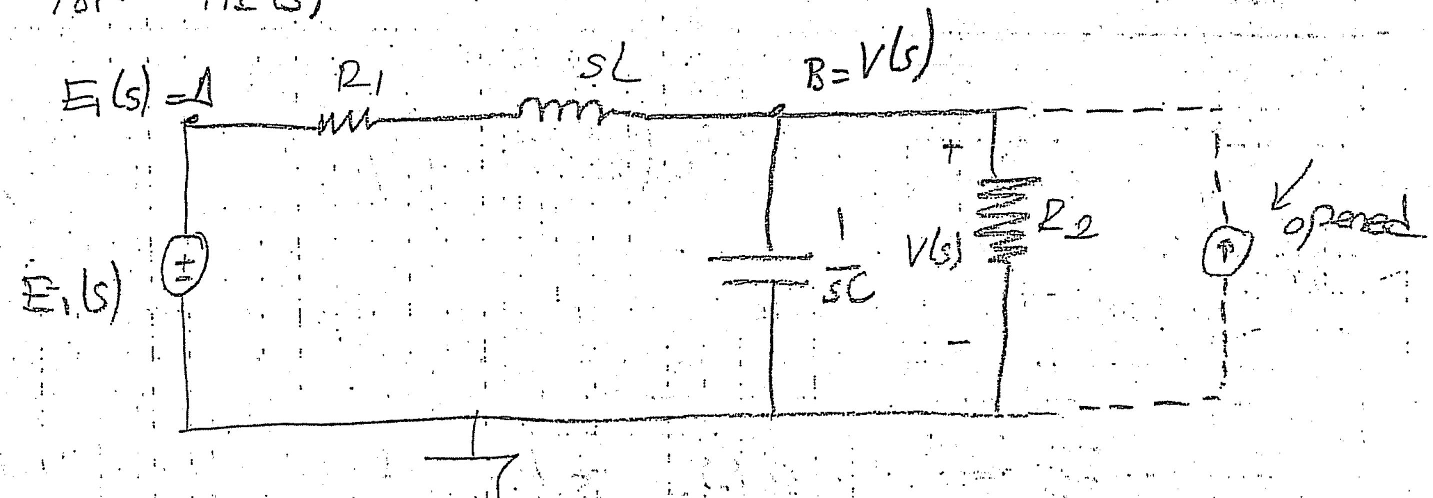
$$H_1(s) = \frac{V(s)}{E_1(s)}\Bigg|_{I_2(s)=0} \qquad H_2(s) = \frac{V(s)}{I_2(s)}\Bigg|_{E_1(s)=0}$$

$I_2(s)=0$ ↳ open circuit

$E_1(s)=0$ ↳ short circuit

in this question initial conditions are assumed to be equal to 0.

For $H_1(s)$



$$\cancel{E_1(s) - V(s)} = R_1 sC V(s) + V(s) + V_s$$

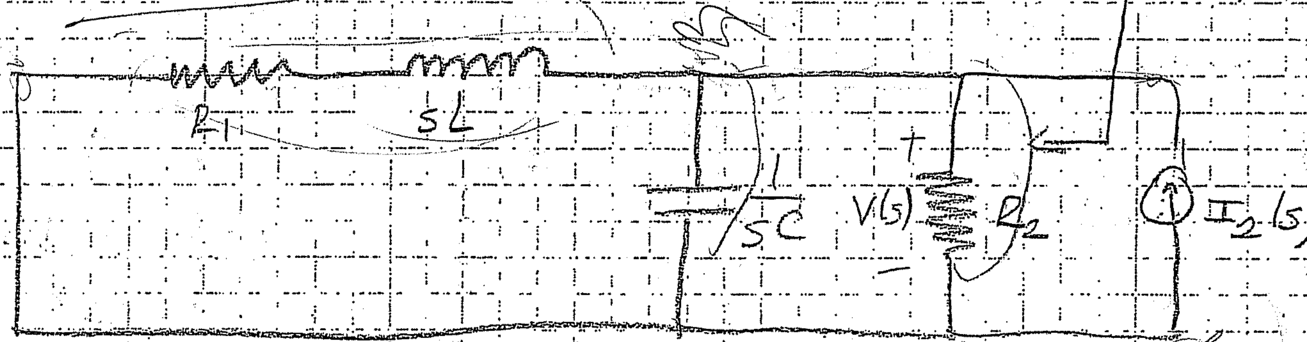
$$E_1(s) = (2 + R_1 sC)\frac{V_s}{E_1(s)}$$

$$\frac{E_1(s) - V(s)}{R_1 + sL} = \frac{V(s)}{\frac{1}{sC}} + \frac{V(s)}{R_2} \qquad \frac{1}{(2+R_2 sC)}$$

$$\left.\frac{V(s)}{E_1(s)}\right|_{I_2(s)=0} = H_1(s) = \frac{1}{1+(R_1+sL)\left(\frac{1}{R_2}+sC\right)}$$

$$1 + \frac{R_1+sL}{R_2} + R_1 s^2 CL$$

For H_2



$$V(s) = I(s)\,Z_{in}(s) \qquad Z_{in}(s) = \left.\frac{V(s)}{I(s)}\right|_{E_1(s)=0} = H_2(s)$$

$$\frac{1}{Z_{in}(s)} = \frac{1}{R_1+sL} + \frac{1}{\frac{1}{sC}} + \frac{1}{R_2}$$

(R_2) $\quad (R_1+sL)R_2$ $\quad (R_1+sL)$

$$\frac{1}{Z_{in}(s)} = \frac{R_2 + sC(R_1+sL)R_2 + R_1+sL}{R_2(R_1+sL)}$$

$$Z_{in}(s) = \frac{R_1R_2 + sLR_2}{s^2CLR_2 + s(CR_1R_2+L) + R_1+R_2}$$

Phasor domain is used for a specific frequency whereas Laplace domain is used for all frequencies

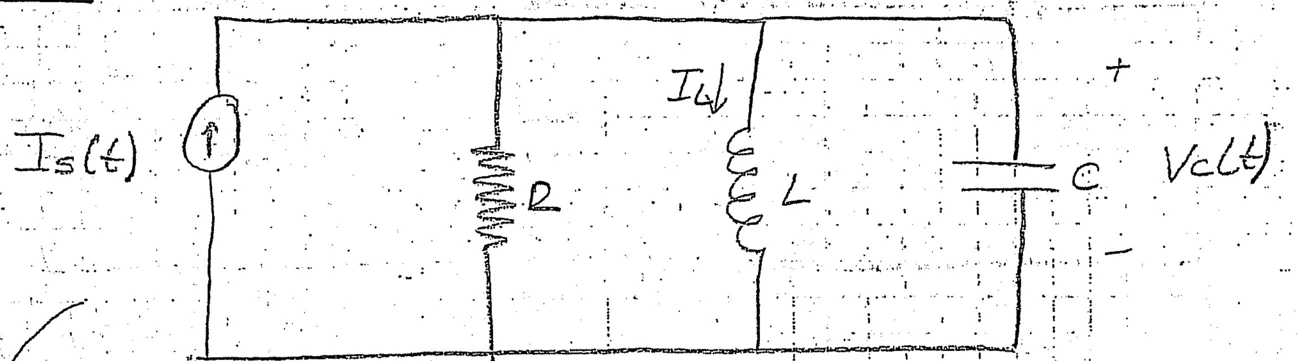
For s-domain analysis: superposition theorem to find the response as the sum of

a) zero-input response (all inputs are turned off)

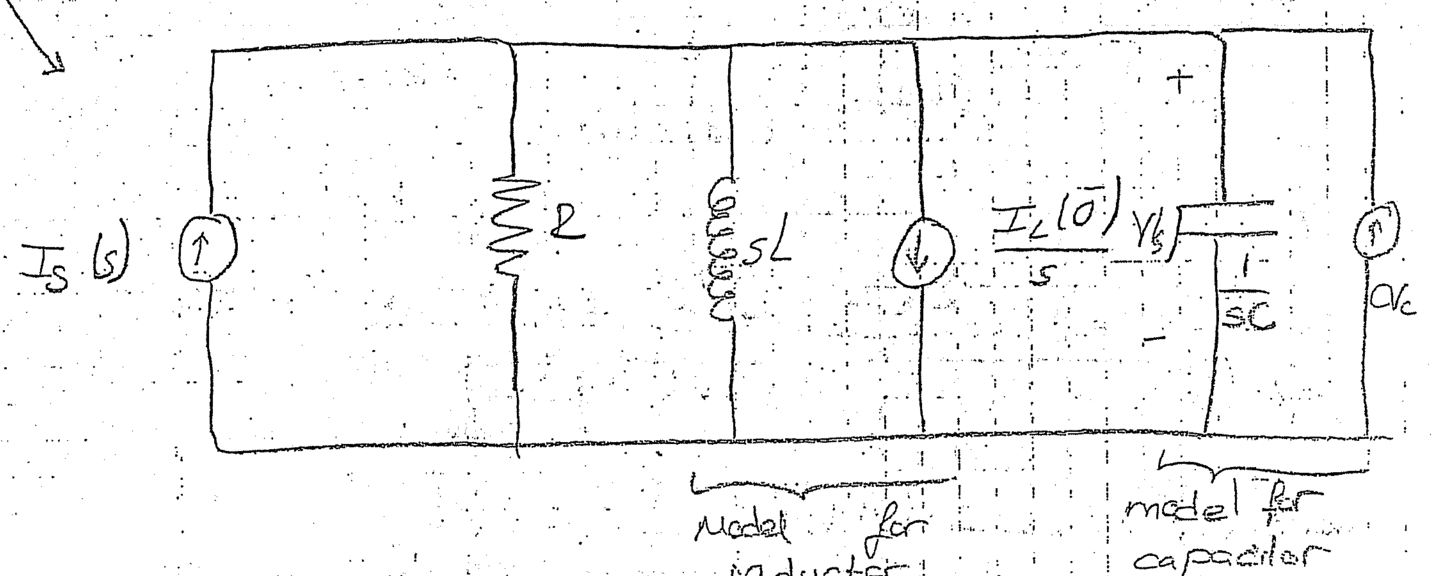
b) zero-state response (all initial conditions are assumed to be equal to 0)

$$Y(s) = Y_{zs}(s) + Y_{zi}(s)$$

Ex:



$V_c(0^-) = V_0 \quad I_L(0^-) = I_0$



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$V(s)$ can be written as the superposition of these 3 sources (2 of them obtained from initial conditions, 1 of them obtained from real source $I_s(s)$)

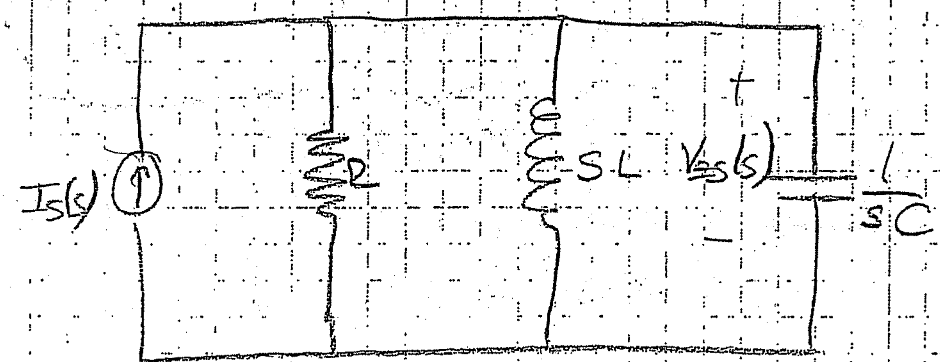
$\int_0$

$$V_c(s) = V_{2s}(s) + V_{2i}(s)$$

due to $I_L(0^-)$ and $V_c(0^-)$

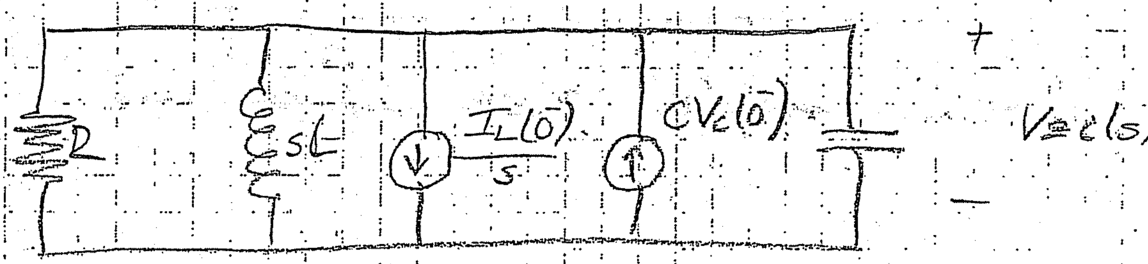
due to $I_s(s)$

For $V_{2s}(s)$: (turn off sources due to initial conditions)



$$V_{2s}(s) = \frac{sRL}{s^2LRC + sL + R} I_s(s)$$

For $V_{2i}(s)$: (turn of the real source)



$$V_{zi_1}(s) = \frac{-RLs}{s^2RLC+sL+R} \underbrace{\frac{I_L(0^-)}{s}}_{\text{source due to initial condition}}$$

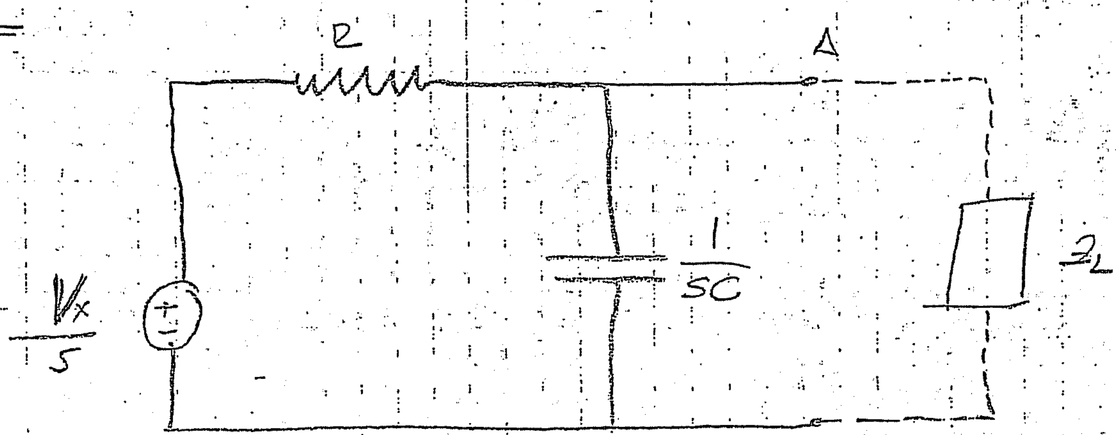
$$V_{zi_2}(s) = \frac{RLs}{s^2RLC+sL+R} \underbrace{C\,V_c(0^-)}_{\text{due to second initial condition}}$$

$$V_{total}(s) = V_{zs}(s) + V_{zi_1}(s) + V_{zi_2}(s)$$

$$= \underbrace{\frac{RLs\,I(s)}{s^2+RLC+sL+R}}_{\text{zero-state}} + \underbrace{\frac{RLs}{s^2RLC+sL+R}\left(CV_c(0^-) - \frac{I_L(0^-)}{s}\right)}_{\text{zero-input}}$$

Thevenin – norton valid in Laplace

Ex:



Find thevenin equivalent circuit between A-B

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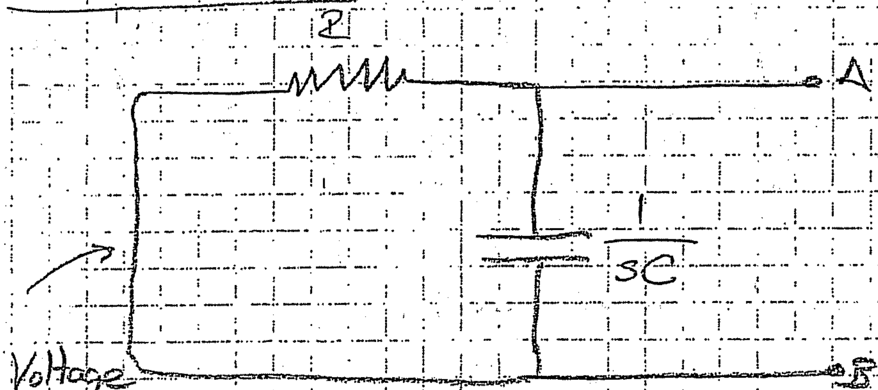
For Z_{TH}

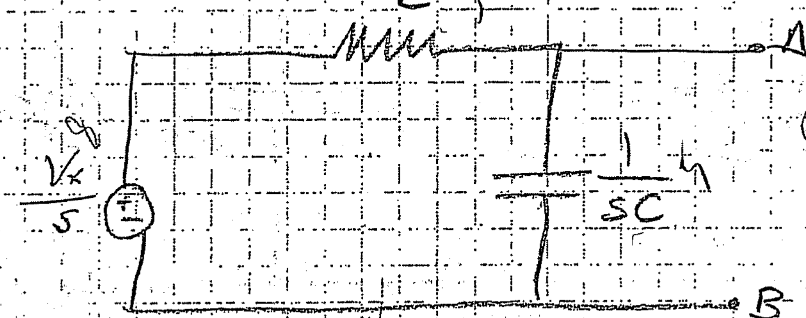


Voltage source is shorted

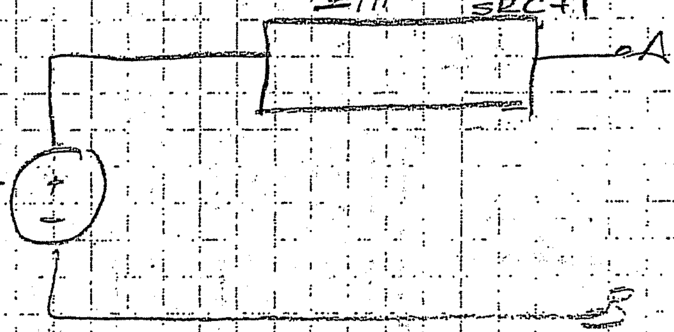
$$Z_{TH} = \frac{R \times \frac{1}{sC}}{R + \frac{1}{sC}}$$

$$Z_{TH} = \frac{R}{sRC+1}$$

For V_{TH}



$$V_{AB} = V_{TH} = \frac{V_x}{s} \quad \frac{1/sC}{R + \frac{1}{sC}} \Rightarrow \boxed{V_{TH} = \frac{V_x}{s} \quad \frac{1}{sRC+1}}$$



$$V_{TH} = \frac{V_x}{s} \times \frac{1}{sRC+1}$$

Thevenin equivalent circuit

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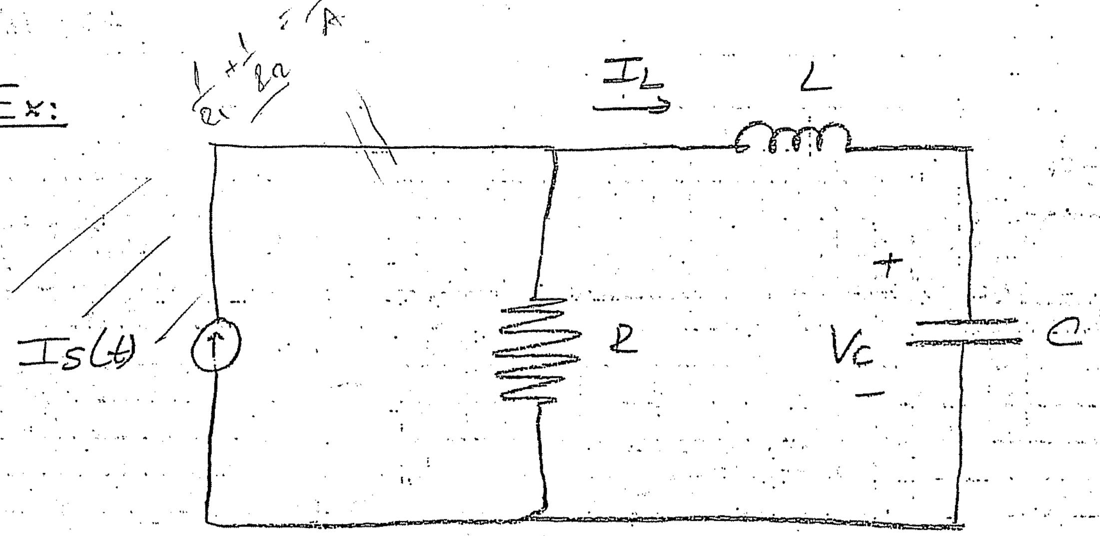
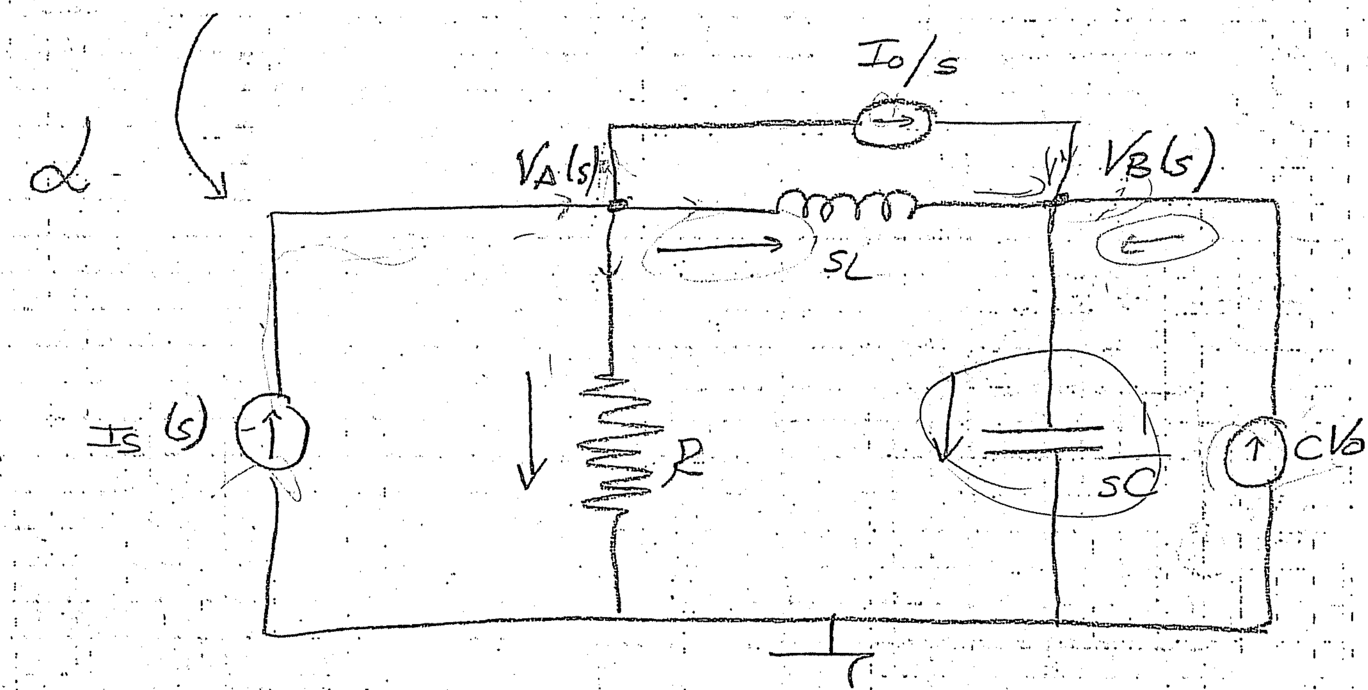
Ex:



$I_L(0^-) = I_0$

$V_C(0^-) = V_0$



Node A: $$I_s(s) = \frac{V_A(s)}{R} + \frac{V_A(s) - V_B(s)}{sL} + \frac{I_0}{s}$$

Node B: $$\frac{V_A(s) - V_B(s)}{sL} + \frac{I_0}{s} + CV_0 = \frac{V_B(s)}{\frac{1}{sC}}$$

$$\begin{array}{cc} V_A(s) & V_B(s) \\ \end{array}$$

$$\begin{bmatrix} \frac{1}{R} + \frac{1}{sL} & \frac{-1}{sL} \\ -\frac{1}{sL} & sC + \frac{1}{sL} \end{bmatrix} \begin{bmatrix} V_A(s) \\ V_B(s) \end{bmatrix} = \begin{bmatrix} I_s(s) - \frac{I_0}{s} \\ \frac{I_0}{s} + CV_0 \end{bmatrix}$$

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$$\begin{bmatrix} V_A(s) \\ V_B(s) \end{bmatrix} = K^{-1} M$$

<u>Notes:</u>

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad X^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

$$X^{-1} = \frac{\operatorname{adj}(X)}{\det(X)} \qquad X = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$X^{-1} = \frac{\begin{bmatrix} (-1)^{1+1}\begin{vmatrix} e & f \\ h & i \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} d & f \\ g & i \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} b & c \\ h & i \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} a & c \\ g & i \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} a & b \\ c & h \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} b & c \\ e & f \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} a & c \\ d & f \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^{T}}{\det(X)}$$

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$$K^{-1} = \frac{\begin{bmatrix} sC + \frac{1}{sL} & \frac{1}{sL} \\ 1/sL & \frac{1}{R} + \frac{1}{sL} \end{bmatrix}}{\left(\frac{1}{R} + \frac{1}{sL}\right)\left(sC + \frac{1}{sL}\right) - \left(\frac{-1}{sL}\right)\left(\frac{-1}{sL}\right)}$$

$$K^{-1} = \frac{\begin{bmatrix} sC + \frac{1}{sL} & \frac{1}{sL} \\ \frac{1}{sL} & \frac{1}{R} + \frac{1}{sL} \end{bmatrix}}{\frac{sC}{R} + \cancel{\frac{1}{s^2L^2}} + \frac{1}{sLR} + \frac{C}{L} - \cancel{\frac{1}{s^2L^2}}}$$

$$V_A = \underbrace{\frac{(LCs^2+1)\ I_s(s)}{\frac{1}{R}LCs^2 + Cs + \frac{1}{R}}}_{\text{zero-state}} + \underbrace{\frac{-LCs\ \ I_0 + CV_0}{\frac{1}{R}LCs^2 + Cs + \frac{1}{R}}}_{\text{zero-input}}$$

<u>For zero input response</u>

* If $V_0 = 0$; $I_0 = 0$ the $V_{A_{zi}}(s) = 0 \xrightarrow{\mathcal{L}^{-1}} V_{A_{zi}}(t) = 0$

* by choosing suitable initial conditions

$$V_{A_{zi}}(s)\ \frac{K\cancel{(s+a)}}{(s+b)\cancel{(s+a)}} = \frac{K}{s+b} \xrightarrow{\mathcal{L}^{-1}} Ke^{-bt}u(t)$$

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* Initial conditions can be chosen such that some natural frequencies are not observable at output.

In general:

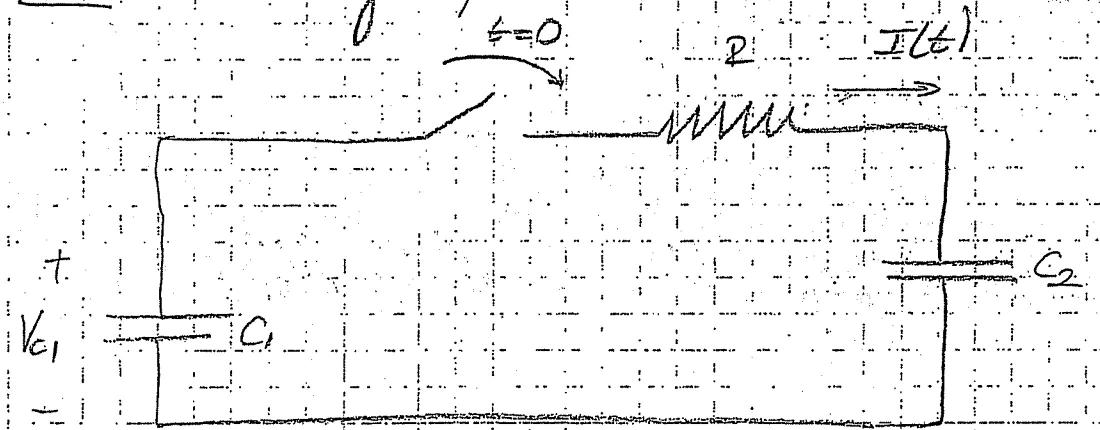
$$V_x(s) = \frac{\Delta_x(s)}{\Delta(s)}$$

→ Total response

* The roots of $\Delta(s)$ are natural poles.

* The poles of $\Delta x(s)$ are the forced poles due to input and they create the forced response.

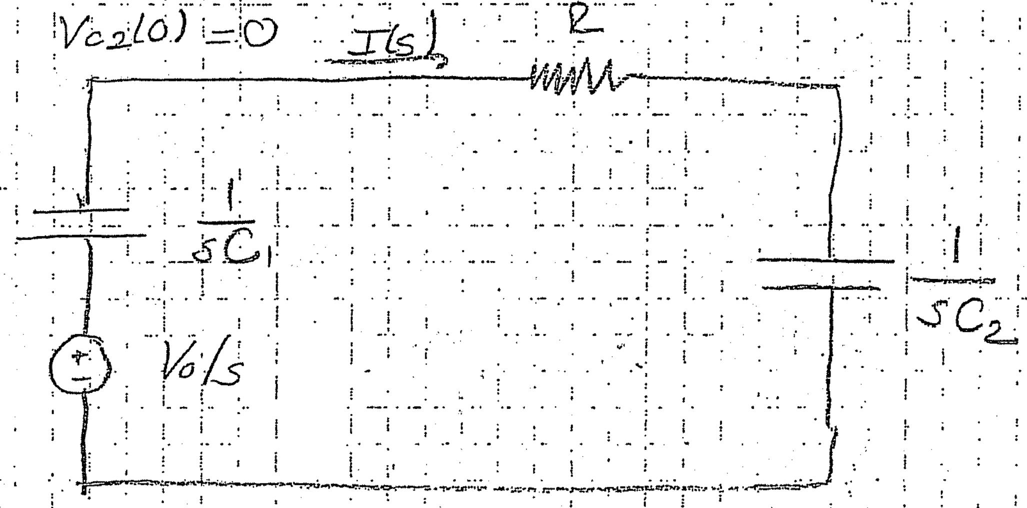
Ex: Switching operation



$\mathcal{L}$

$V_{c1}(0) = V_0$

$V_{c2}(0) = 0$



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Osman Kılıç
2 Adet

$$= \frac{V_0/R}{s + \frac{1}{RC_e}} \qquad C_e = \frac{C_1 C_2}{C_1 + C_2}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$I(t) = \frac{V_0}{R} e^{-\frac{t}{RC_e}} u(t)$$

If $R \longrightarrow 0$

$$I(s) = \frac{V_0/s}{0 + \frac{1}{sC_1} + \frac{1}{sC_2}} = \frac{V_0}{\frac{C_1 C_2}{C_1 + C_2}}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$I(t) = \frac{V_0 \; C_1 + C_2}{C_1 \cdot C_2} \delta(t) = V_0 C_e \delta(t)$$

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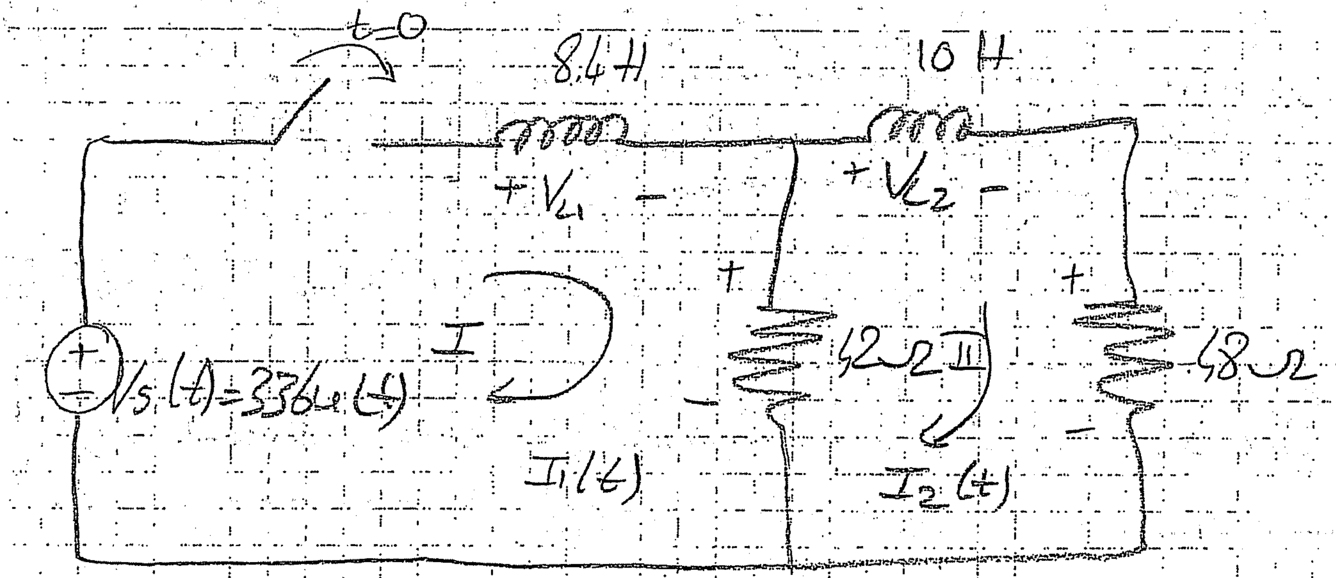

$I_1(t) = ?$ $I_2(t) = ?$

Mesh I

$$-V_s(t) + V_{L1} + V_{42\Omega} = 0$$

$$-V_s(t) + L_1 \frac{dI_1(t)}{dt} + 42\left(I_1(t) - I_2(t)\right) = 0$$

$$\mathcal{L} \quad -V_s(s) + \underbrace{L_1}_{8.4\,H}\left[sI_1(s) - I_1(0^-)\right] + 42\left[I_1(s) - I_2(s)\right] = 0$$

Mesh II

$$V_{L2} + V_{48\Omega} - V_{42\Omega} = 0$$

$$L_2 \frac{dI_2}{dt} + 48 I_2 + 42(I_2 - I_1) = 0$$

$$\mathcal{L} \quad \underbrace{L_2}_{10\,H}\left[sI_2(s) - I_2(0^-)\right] + 48 I_2(s) + 42\left(I_2(s) - I_1(s)\right) = 0$$

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$$10s\,I_2(s) - 10\,I_2(0^-) + 48\,I_2(s) + 42\,I_2(s) - 42\,I_1(s) = 0$$

$$\boxed{(90 + 10s)\,I_2(s) - 42\,I_1(s) = 10\,I_2(0^-)}$$

$$-V_s(s) + 8.4s\,I_1(s) - 8.4\,I_1(0^-) + 42\,I_1(s) - 42\,I_2(s) =$$

$$\boxed{(8.4s + 42)\,I_1(s) - 42\,I_2(s) = V_s(s) + 8.4\,I_1(0^-)}$$

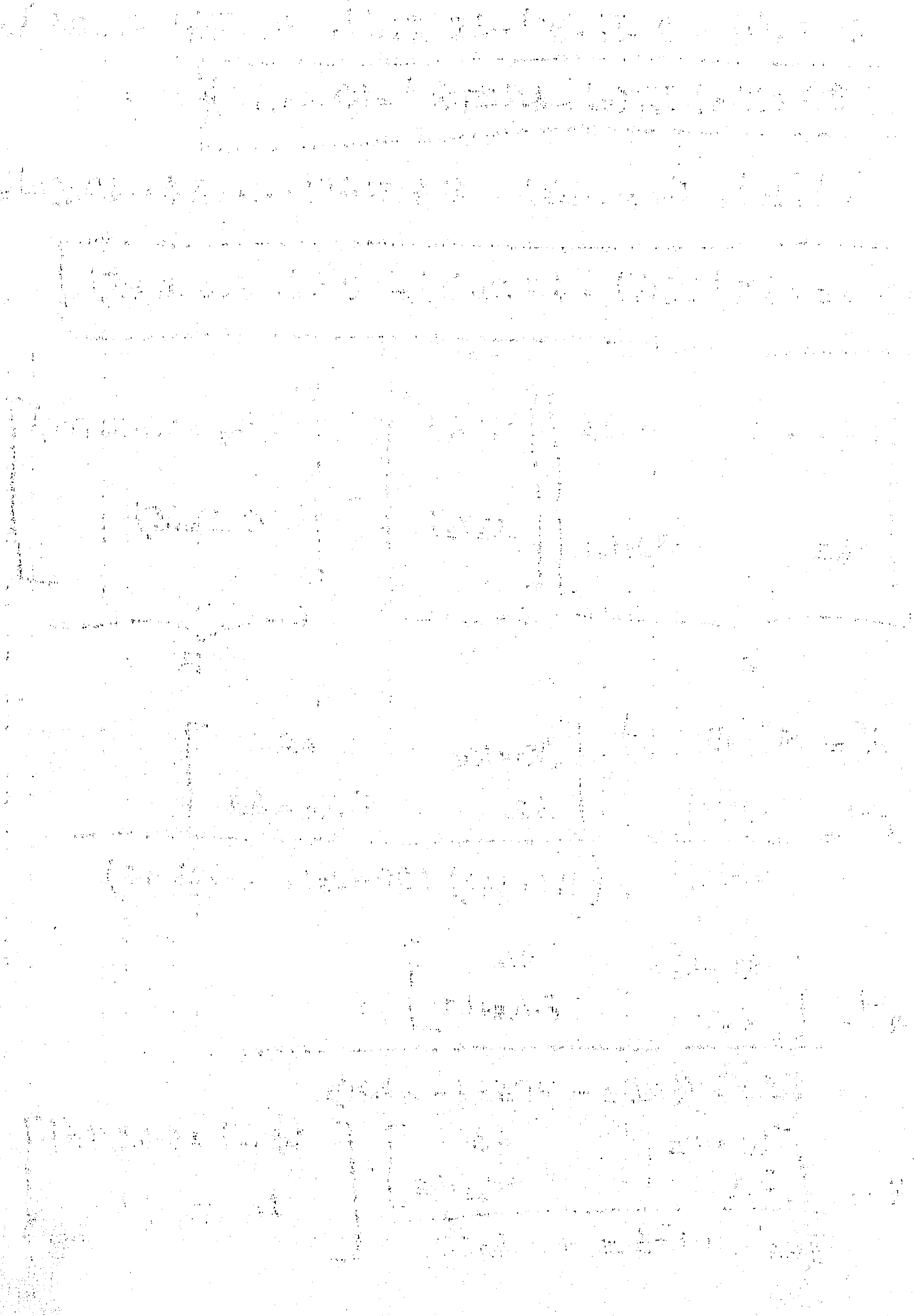
$$\underbrace{\begin{bmatrix} 8.4s+42 & -42 \\ -42 & 90+10s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} V_s(s) + 8.4\,I_1(0^-) \\ 10\,I_2(0^-) \end{bmatrix}}_{B}$$

$$X = A^{-1}B$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{\begin{bmatrix} 90+10s & 42 \\ 42 & 8.4s+42 \end{bmatrix}}{(8.4s+42)(90+10s) - (-42)(-42)}$$

$$A^{-1} = \frac{\begin{bmatrix} 90+10s & 42 \\ 42 & 8.4s+42 \end{bmatrix}}{84s^2 + (420s - 756s) - 4320}$$

$$X = \frac{\begin{bmatrix} 90+10s & 42 \\ 42 & 8.4s+42 \end{bmatrix}}{84s^2 + 1.176s + 4320} \begin{bmatrix} V_s(s) + 8.4\,I_1(0^-) \\ 10\,I_2(0^-) \end{bmatrix}$$



$$10s\,I_2(s) - 10\,I_2(0^-) + 48\,I_2(s) + 42\,I_2(s) - 42\,I_1(s) = 0$$

$$\boxed{(90 + 10s)\,I_2(s) - 42\,I_1(s) = 10\,I_2(0^-)}$$

$$-V_s(s) + 8.4s\,I_1(s) - 8.4\,I_1(0^-) + 42\,I_1(s) - 42\,I_2(s) =$$

$$\boxed{(8.4s + 42)\,I_1(s) - 42\,I_2(s) = V_s(s) + 8.4\,I_1(0^-)}$$

$$\underbrace{\begin{bmatrix} 8.4s+42 & -42 \\ -42 & 90+10s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} V_s(s) + 8.4\,I_1(0^-) \\ 10\,I_2(0^-) \end{bmatrix}}_{B}$$

$$X = A^{-1} B$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{\begin{bmatrix} 90+10s & 42 \\ 42 & 8.4s+42 \end{bmatrix}}{(8.4s+42)(90+10s) - (-42)(-42)}$$

$$A^{-1} = \frac{\begin{bmatrix} 90+10s & 42 \\ 42 & 8.4s+42 \end{bmatrix}}{84s^2 + (420s - 756s) - 4320}$$

$$X = \frac{\begin{bmatrix} 90+10s & 42 \\ 42 & 8.4s+42 \end{bmatrix}}{84s^2 + 1.176s + 4320} \begin{bmatrix} V_s(s) + 8.4\,I_1(0^-) \\ 10\,I_2(0^-) \end{bmatrix}$$

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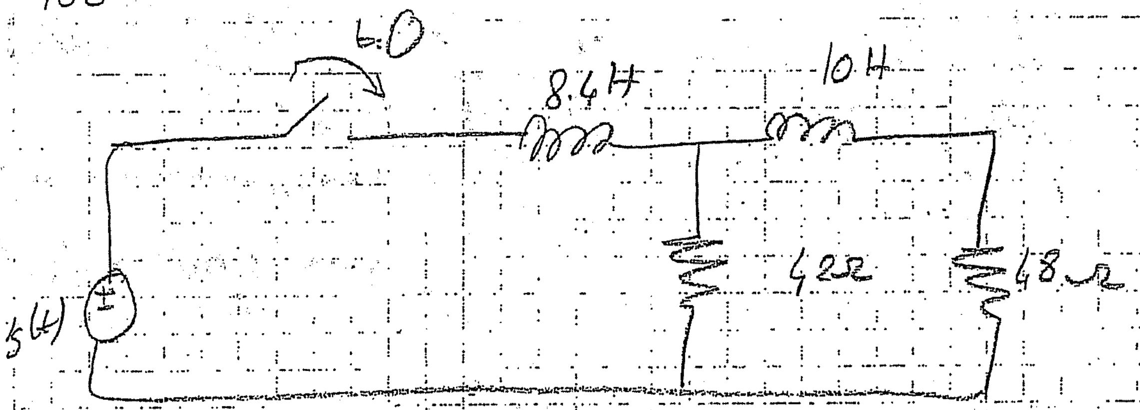

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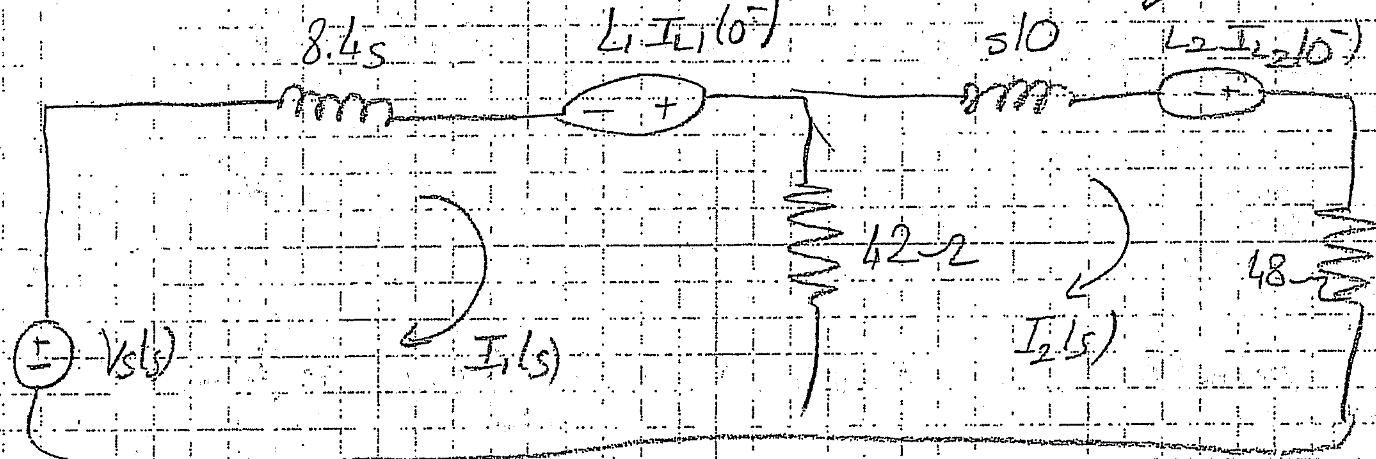

$I_{L_1}(0^-) = I_1(0^-)$ $\qquad$ $I_{L_2}(0^-) = I_2(0^-)$

$$-V_s(s) + (8.4s)I_1(s) - L_1 I_1(0^-) + 42(I_1(s) - I_2(s)) = 0$$

$$s10\, I_2(s) - L_2 I_2(0^-) + 42(I_2(s) - I_1(s)) + 48 I_2(s) = 0$$

$$I_1(s) = \frac{(s10+90)V_s(s)}{84(s+2)(s+12)} + \frac{84(s+9)I_1(0^-) + 42 I_2(0^-)}{84(s+2)(s+12)}$$

Find any initial condition such that only one mode (either $s=-2$ or $s=-12$) is excited at the output of zero input response

$I_1(0^-) = K I_2(0^-)$

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$$\frac{N(s)}{D(s)} = \frac{M\cancel{(s+12)}}{84\cancel{(s+12)}(s+2)} = \frac{M}{84(s+2)} \xrightarrow{\mathcal{L}^{-1}} \underbrace{\frac{M}{84} e^{-2t} u(t)}_{\text{zero input output}}$$

$$84(s+9)\,I_1(0^-) + 42\,I_2(0^-) = M(s+12)$$

$$42\left[2s + 18\,I_1(0^-) + I_2(0^-)\right] = M(s+12)$$

$$84\left[s + 9\,I_1(0^-) + \frac{I_2(0^-)}{2}\right] = M(s+12)$$

$$M = 84 \qquad s + 9\,I_1(0^-) + \frac{I_2(0^-)}{2} = s + 12$$

$$9\,I_1(0^-) + \frac{I_2(0^-)}{2} = 12$$

$$I_1(0^-) = 0 \implies I_2(0^-) = 24$$

$I_1(0^-) = 0 \quad I_2(0^-) = 0$ but $V_s(t) \neq 336\,u(t)$

$$V_s(t) = \sin(3t) \xrightarrow{\mathcal{L}} \frac{3}{s^2+9}$$

$$I_{2\text{-}s}(s) = \frac{(10s+90)\,V_s(s)}{84\,(s+2)(s+12)}$$

$$= \frac{10(s+9)\,\cancel{3}}{(s+2)(s+12)(s^2+9)\,\cancel{84}_{14}} = \frac{s+9}{(s+2)(s+12)(s^2+9)}$$

$$\mathcal{L}^{-1} = \frac{A}{s+2} + \frac{B}{s+12} + \frac{C}{s-3i} + \frac{C^*}{s+3i}$$

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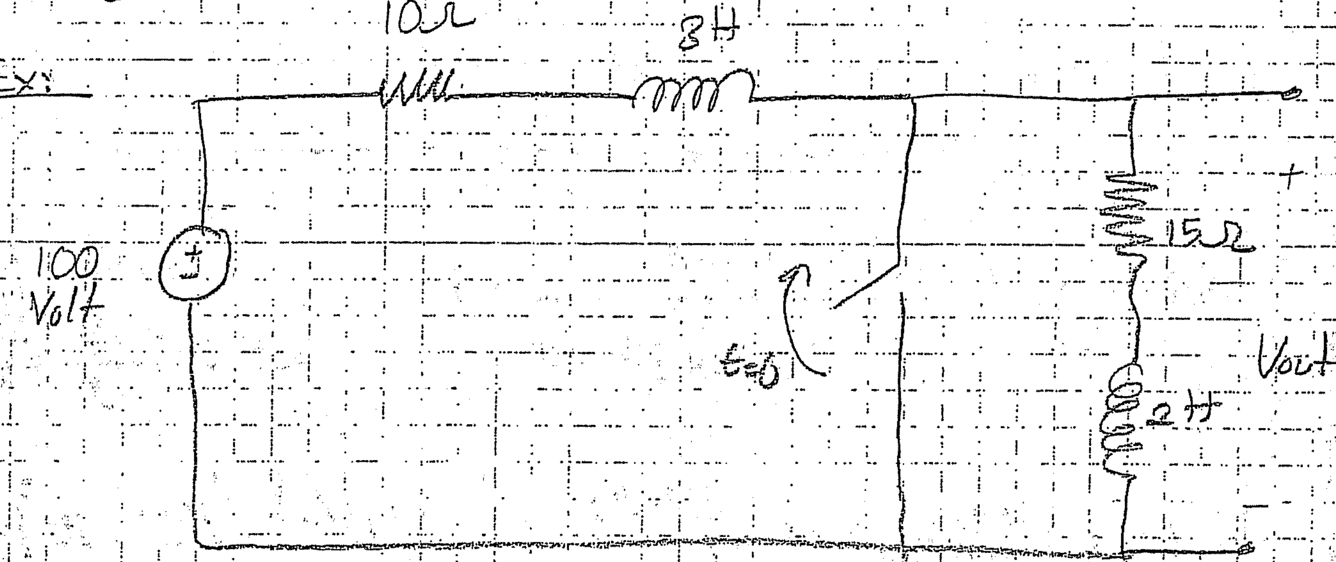
$$\underbrace{Ae^{-2t}u(t) + Be^{-12t}u(t)}_{\text{transient part}} + \underbrace{2|C|\cos(3t+\theta)u(t)}_{SSS}$$

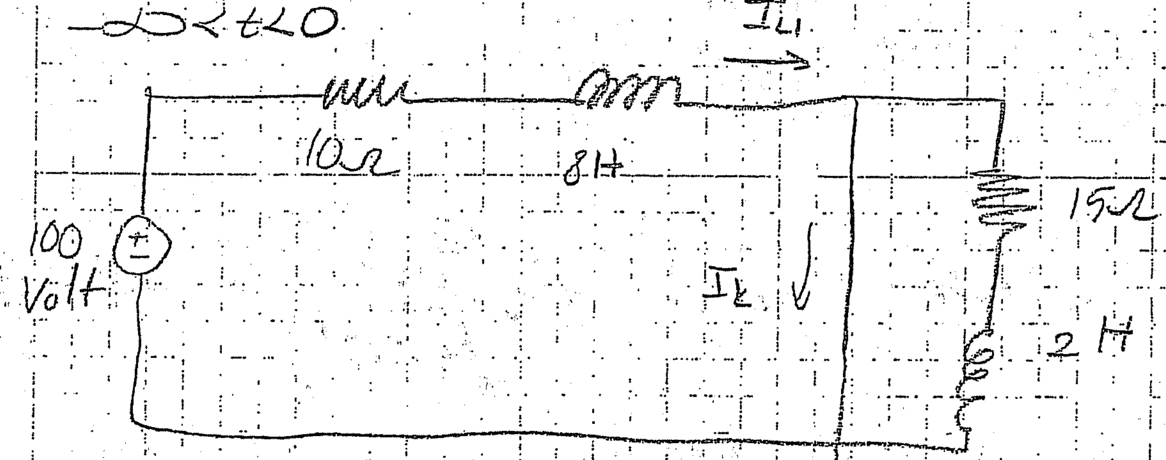
$\theta = \angle C$

Ex:



Find V_{out} $t \geq 0$

$-\infty < t < 0$



$I_2(0^-) = 0$

$I_1(0^-) = I_L(0^-)$

$I_1(0^-) = \frac{100}{10} = 10A$

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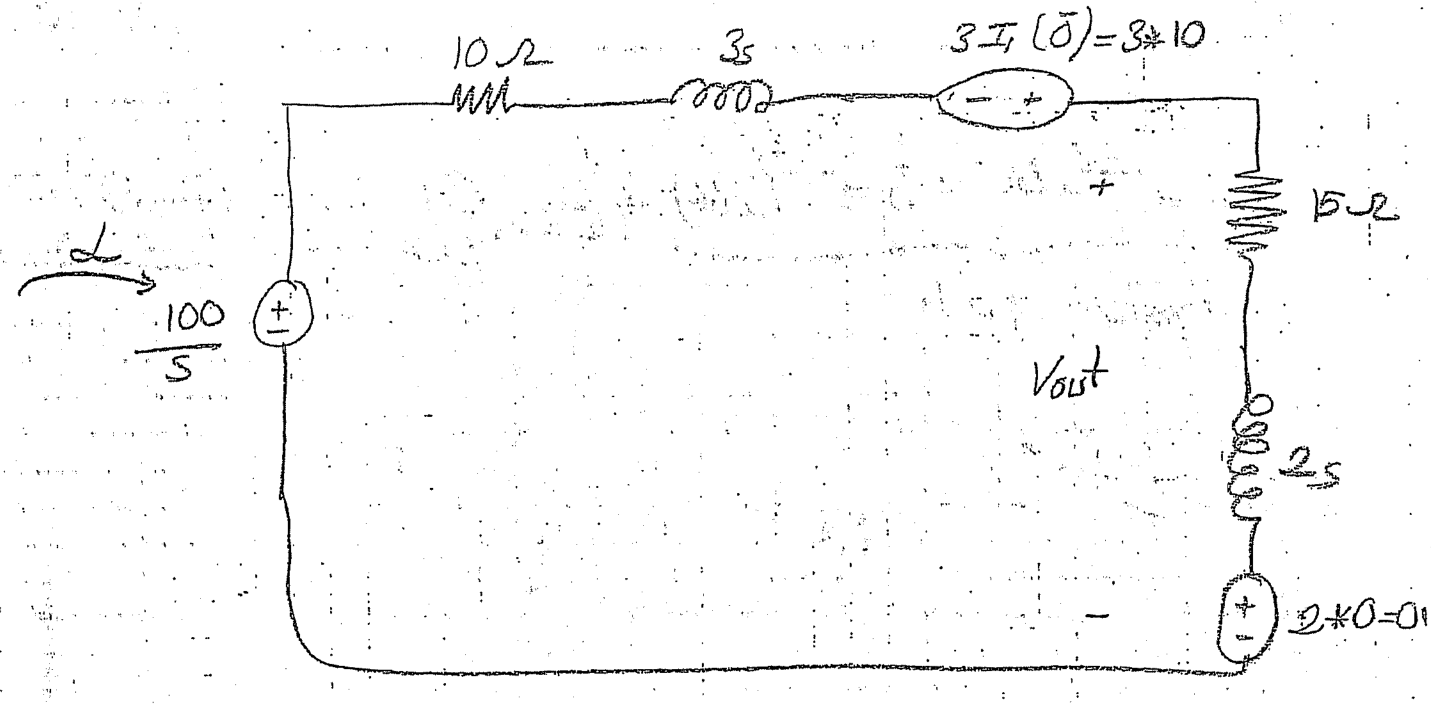


$$V_{out} = \left(\frac{100}{s}+30\right)\frac{15+2s}{15+2s+10+3s} \Rightarrow$$

$$\begin{array}{r|l} 2s+15 & 5s+25 \\ -2s+10 & \frac{2}{5} \\ \hline 5 & \end{array}$$

$$V_{out}(s) = \left(\frac{100}{s}+30\right)\frac{15+2s}{25+5s} \qquad V_{out}(s) = \left(\frac{100}{s}+30\right)\left(\frac{2}{5}+\frac{5}{5s+25}\right)$$

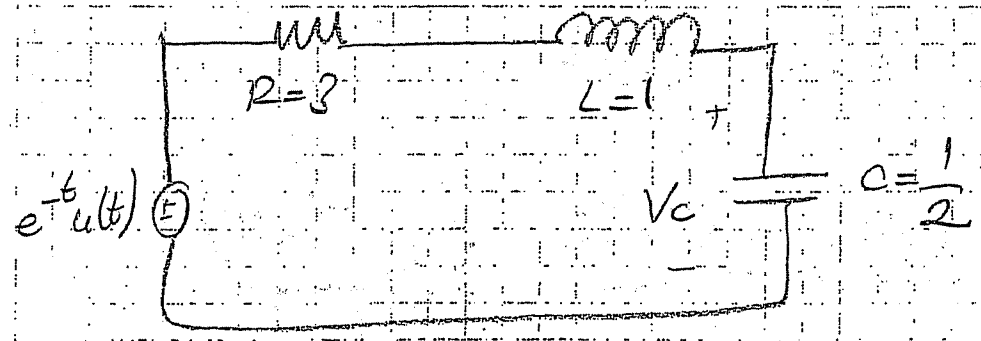
$$V_{out}(s) = \frac{40}{s} + 12 - \frac{100}{s(s+5)}$$

$\mathcal{L}^{-1}$

$$V_{out}(t) = 40u(t) + 12\delta(t) - \mathcal{L}^{-1}\left\{\frac{100}{s(s+5)}\right\} = \frac{K_1}{s}+\frac{K_2}{s+5}$$

$$100 = K_1(s+5)+sK_2 \quad K_1=20 \text{ and } K_2=-20$$

$$40u(t)+12\delta(t) - \mathcal{L}^{-1}\left\{\frac{20}{s}+\frac{-20}{s+5}\right\} = 60u(t)+12\delta(t)-2\ldots$$



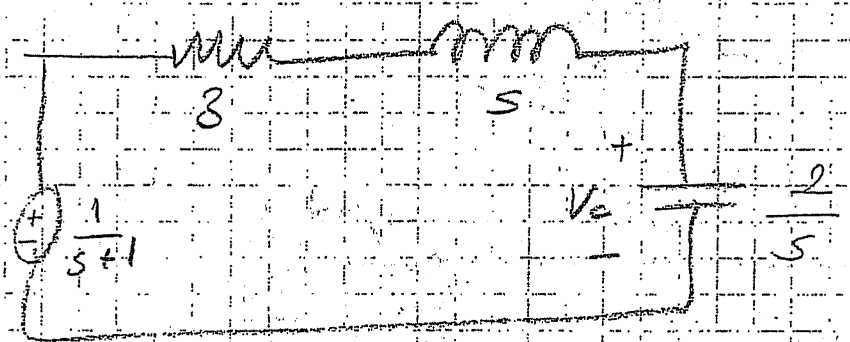
$I_L(0^-)=0$
$V_c(0^-)=0$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 3s + 2 = 0$$

$$(s+1)(s+2) = 0$$

$\mathcal{L}$



$$V_c(s) = \frac{\frac{2}{s}}{\frac{2}{s}+s+3} \cdot \frac{1}{s+1}$$

$$V_c(s) = \frac{2}{s^2+3s+2} \cdot \frac{1}{s+1} = \frac{2}{(s+1)^2(s+2)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$2 = A(s+1)^2 + B(s+1)(s+2) + C(s+2)$$

$$2 = (A+B)s^2 + (2A+3B+C)s + A + 2B + 2C$$

$A + B = 0 \implies A = -B$ $\quad$ $A = 2$

$2A + 3B + C = 0 \implies B + C = 0$ $\quad$ $B = -2$

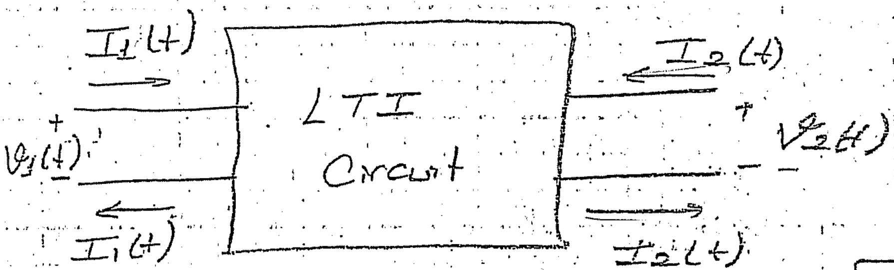
$A + 2B + 2C = 2$ $\quad$ $C = 2$

$$V_c(s) = \frac{2}{s+2} + \frac{-2}{(s+1)} + \frac{2}{(s+1)^2}$$

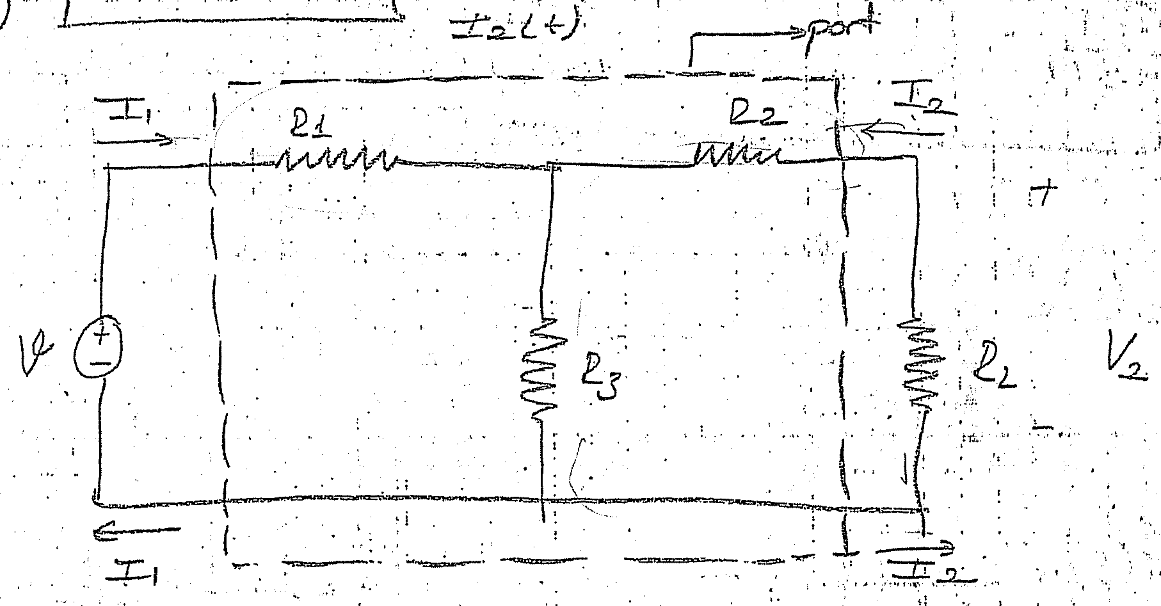
$$V_c(t) = \underbrace{2e^{-2t}u(t) - 2e^{-t}u(t)}_{\text{homogeneous}} + \underbrace{2te^{-t}u(t)}_{\text{particular}}$$

Two Ports:

$$f(v_1(t);\ v_2(t);\ I_1(t),\ I_2(t);\ t) = 0$$



Ex:



$I_1 = I_1'$ $\quad I_2 = I_2'$

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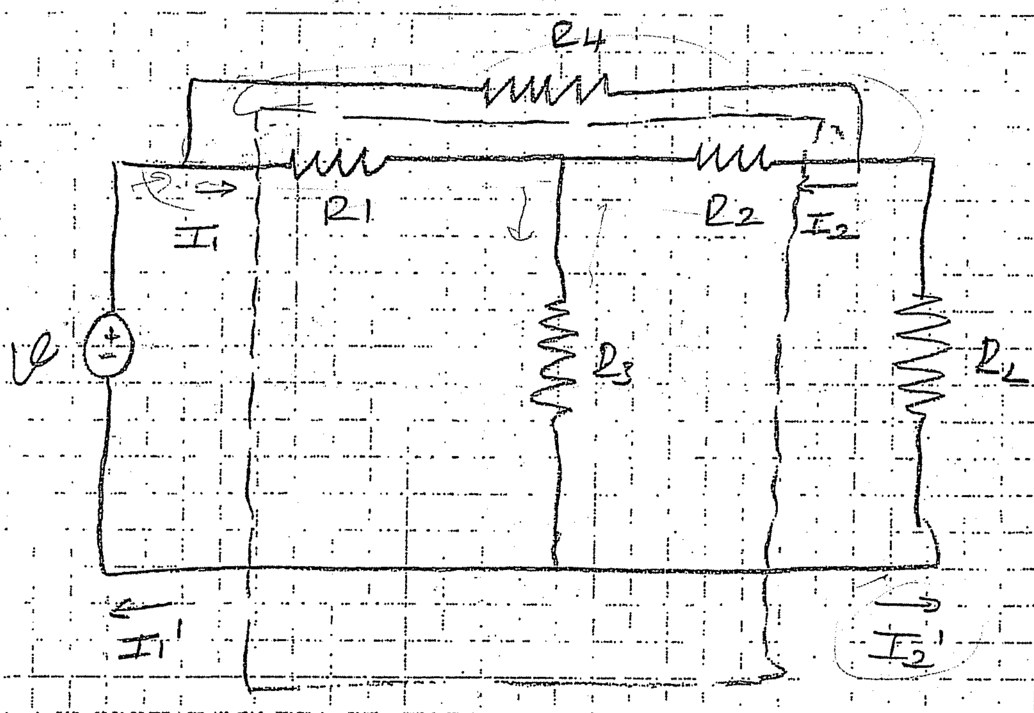

$I_1 \neq I_1'$ $\qquad$ $I_2 = I_2'$

*It is assumed that LTI ports has no independent sources. By using superposition the effect of independent sources could be further analysed.

Resistance Matrix (open-circuit matrix)

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$V_1 = r_{11} I_1 + r_{12} I_2$$

$$V_2 = r_{21} I_1 + r_{22} I_2$$

$$r_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

$$r_{12} = \left.\frac{V_1}{I_2}\right|_{I_2=0}$$

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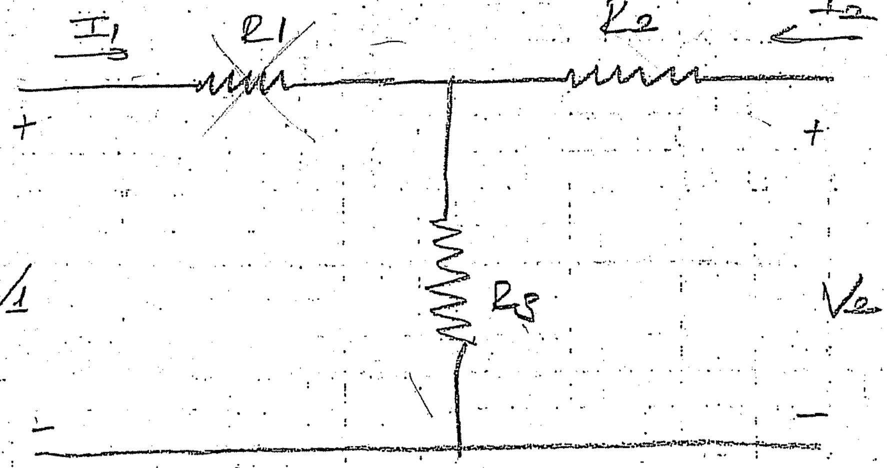
$$r_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \qquad r_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

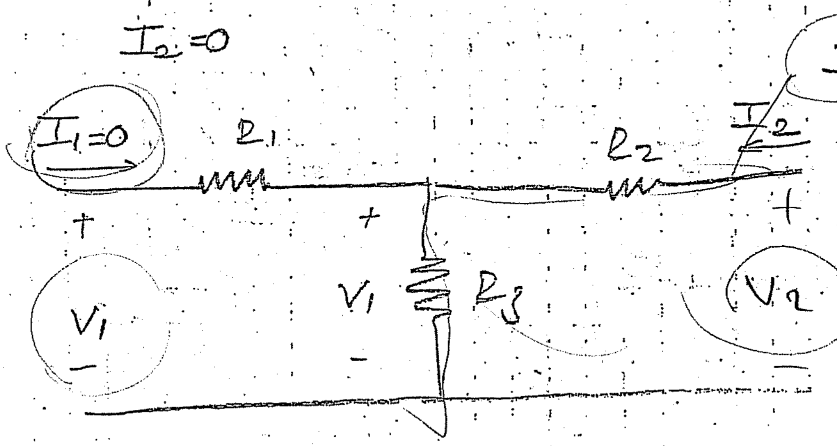
Ex:



$$r_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = R_1 + R_3 \qquad V_1 = R_1 I_1 + R_3 (I_1 + I_2)$$

when $I_2 = 0$

$$r_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = R_1 + R_3 \qquad r_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = R_3$$



$$= \left.\frac{V_1}{I_2}\right|_{I_1=0} = R_3$$

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$$r_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$V_2 = R_2 I_2 + (I_1 + I_2) R_3$$

$$r_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = R_3$$

$$r_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

$$r_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = R_2 + R_3$$

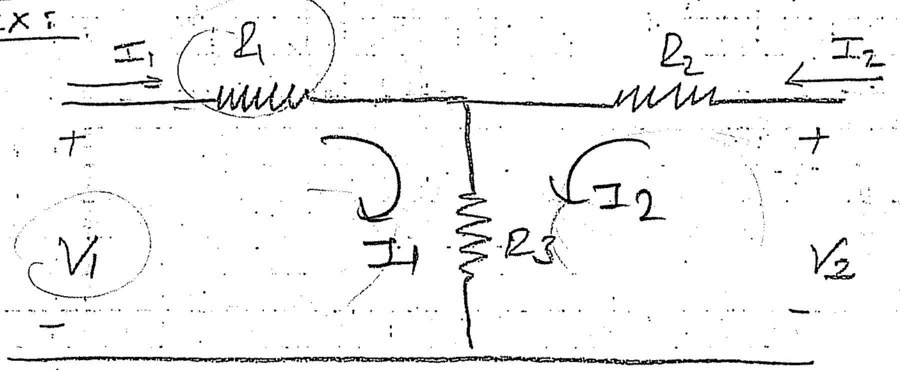
$$\bar{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \bar{R} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Transmition Matrix (ABCD matrix)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Ex:



ABCD ?

$$A = \left.\frac{V_1}{V_2}\right|_{-I_2=0} \qquad B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} \qquad D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$$

$$V_1 = R_1 I_1 + (I_1 + I_2) R_3$$

$$V_2 = R_2 . I_2 + (I_1 + I_2) R_3$$

$$I_2 = 0 \Longrightarrow \left.\begin{array}{l} V_1 = (R_1 + R_3) I_1 \\ V_2 = R_3 I_1 \end{array}\right\} \Rightarrow A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{R_1 + R_3}{R_3}$$

$V_2 = 0$

$$(R_2 + R_3) I_2 = -I_1 R_3 \Rightarrow I_1 = \frac{(R_2 + R_3) I_2}{-R_3}$$

$$V_1 = R_1 \frac{(R_2 + R_3)}{-R_3} I_2 + \left(\frac{(R_2 + R_3) I_2}{-R_3} + I_2 \right) R_3$$

$$V_1 = \left[\frac{R_1 (R_2 + R_3)}{-R_3} - (R_1 + R_3) + R_3 \right] I_2$$

$$V_1 = \left[\frac{-R_1 R_2 - R_1 R_3 - R_1 R_3 - \cancel{R_3^2} + \cancel{R_3^2}}{R_3} \right] I_2$$

$$V_1 = -\left[\frac{+R_1 R_2 + R_1 R_3 + R_2 R_3}{R_3} \right] I_2$$

$$B = \left. \frac{V_1}{-I_2} \right|_{V_2 = 0} = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_3}$$

$$C = \left. \frac{I_1}{V_2} \right|_{I_2 = 0}$$

$$V_1 = R_3 I_2 + (R_1 + R_3) I_1$$

$$V_2 = R_3 I_2 + (R_2 + R_3) I_2$$

$$I_2 = 0 \Rightarrow V_2 = R_3 I_1 \Rightarrow C = \left. \frac{I_1}{V_2} \right|_{I_2 = 0} = \frac{1}{R_3}$$

$$D = \frac{I_1}{-I_2}\Bigg|_{V_2=0}$$

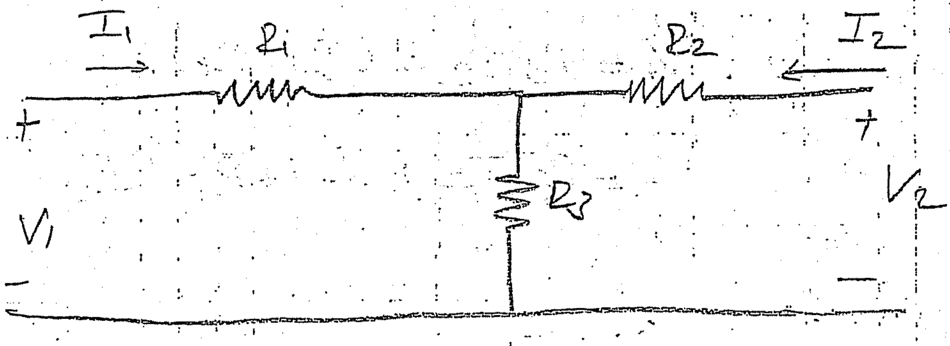
$$V_2 = 0 \Rightarrow (R_2+R_3)\, I_2 + R_3 I_1 = 0$$

$$\frac{I_1}{I_2} = -\frac{R_2+R_3}{R_3} \qquad D = \frac{I_1}{-I_2}\Bigg|_{V_2=0} = \frac{R_2+R_3}{R_3}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \frac{R_1+R_3}{R_3} & \frac{R_1R_2+R_1R_3+R_2R_3}{R_3} \\ \frac{1}{R_3} & \frac{R_2+R_3}{R_3} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Conductance Matrix: (short circuit)

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{\bar{G}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad \bar{G} = \bar{R}^{-1} \qquad \boxed{G = \frac{1}{R}}$$



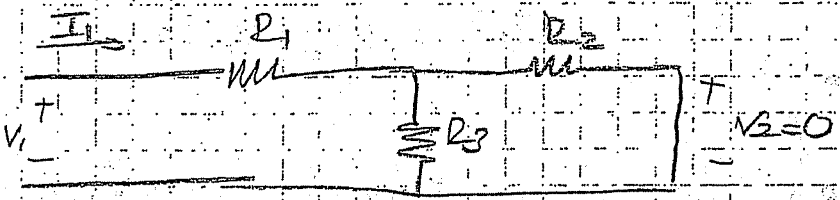
$$V_1 = (R_1+R_3) I_1 + R_3 I_2$$

$$V_2 = R_3 I_1 + (R_2+R_3) I_2$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \qquad g_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

$$g_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \qquad g_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$



$$\left.\frac{V_1}{I_1}\right|_{V_2=0} = R_1 + \frac{R_2 R_3}{R_2+R_3} \qquad \frac{V_1}{I_1} = R_1 + (R_2 \parallel R_3)$$

$$\left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2+R_3}$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3}$$

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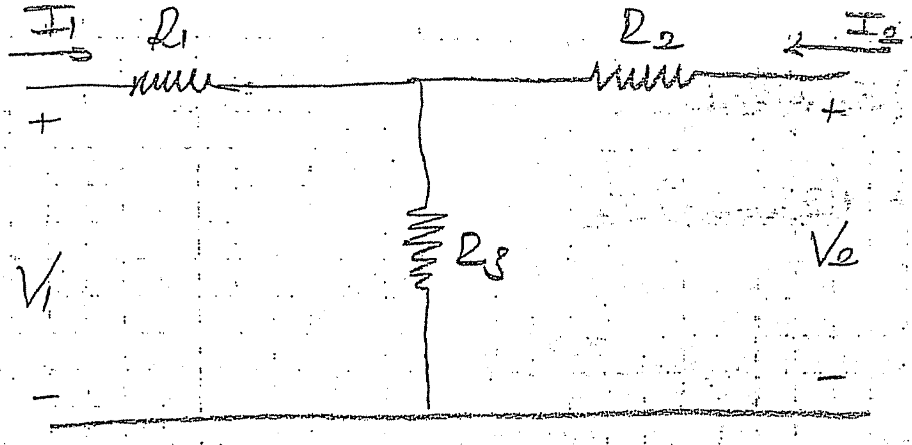
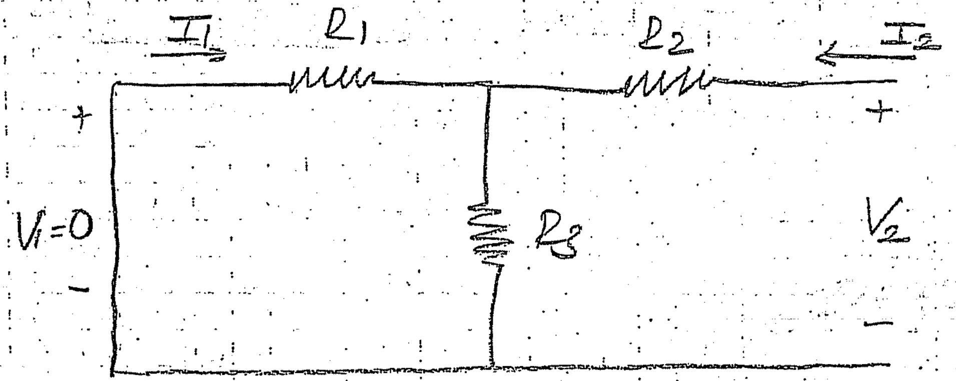


$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$g_{11} = \frac{R_2 + R_3}{R_1R_2 + R_1R_3 + R_2R_3} \qquad g_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$



$$I_1 = I_2 * \frac{-R_3}{R_1+R_3} \Longrightarrow I_2 = I_1 \frac{R_1+R_3}{-R_3}$$

$$\frac{V_2}{I_2} = (R_1 \,//\, R_3) + R_2 = \frac{R_1R_3}{R_1+R_3} + R_2 = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1+R_3}$$

$$V_2 = \frac{R_1R_3 + R_1R_2 + R_2R_3}{R_1 + R_3} I_2$$

$$V_2 = -\frac{R_1R_3 + R_1R_2 + R_2R_3}{R_3} I_1$$

$$\boxed{g_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{-R_3}{R_1R_2 + R_2R_3 + R_1R_3}}$$

HW: $g_{21} \neq g_{12}$

$$g_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

$$V_1 = (R_1 + R_3) I_1 + R_3 I_2$$

$$V_2 = R_3 I_1 + (R_2 + R_3) I_2$$

$$0 = (R_1 + R_3) I_1 + R_3 I_2$$

$$-\frac{R_3 I_2}{R_1 + R_3} = I_1$$

$$V_2 = R_3 \left(\frac{-R_3}{R_1 + R_3} I_2 \right) + (R_2 + R_3) I_2$$

$$V_2 \left[\frac{-\cancel{R_3^2} + R_1R_2 + R_1R_3 + R_2R_3 + \cancel{R_3^2}}{R_1 + R_3} \right] I_2$$

$$g_{22} = \frac{I_2}{V_2}\bigg|_{V_1=0} = \frac{R_1+R_3}{R_1R_2+R_1R_3+R_2R_3}$$

$$\bar{G} = \begin{bmatrix} \dfrac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3} & \dfrac{-R_3}{R_1R_2+R_1R_3+R_2R_3} \\[2ex] \dfrac{-R_3}{R_1R_2+R_1R_3+R_2R_3} & \dfrac{R_1+R_3}{R_1R_2+R_1R_3+R_2R_3} \end{bmatrix}$$

<u>H.W:</u> Show that $\bar{R} = \bar{G}^{-1}$

<u>Hybrid Matrix:</u>

$$H_1 \longrightarrow \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \overbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}^{H} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$H_2 \longrightarrow \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix}}_{H'} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$$H' = H^{-1}$$

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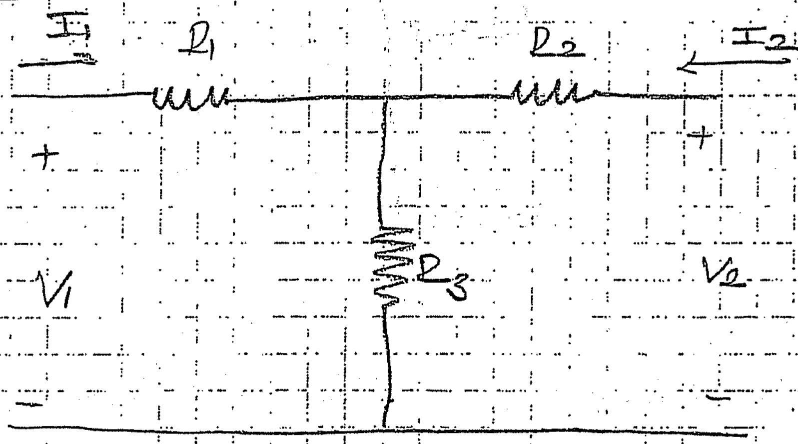
Ex:



$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

$$V_1 = (R_1+R_3)\,I_1 + R_3 I_2$$

$$V_2 = R_3 I_1 + (R_2+R_3)\,I_2$$

$$0 = R_3 I_1 + (R_2+R_3)\,I_2 \Rightarrow I_2 = \frac{-R_3 I_1}{R_2+R_3}$$

$$V_1 = (R_1+R_3)\,I_1 + R_3\left(\frac{-R_3}{R_2+R_3} I_1\right)$$

$$\left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2+R_3} = h_{11} \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{12} = \frac{R_3}{R_2+R_3} \qquad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{-R_3}{R_2+R_3}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{R_2+R_3}$$

$$H = \begin{bmatrix} \dfrac{R_1R_2 + R_1R_3 + R_2R_3}{R_2+R_3} & \dfrac{R_3}{R_2+R_3} \\ \\ \dfrac{-R_3}{R_2+R_3} & \dfrac{1}{R_2+R_3} \end{bmatrix}$$

$$H' = H^{-1} =$$

$$X = \left[\underbrace{\begin{matrix} a & b \\ c & d \end{matrix}}_{A} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \xrightarrow[\text{row operations}]{\text{elementary}} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| A^{-1} \right]$$

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow H^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

$$H^{-1} = \begin{bmatrix} \dfrac{1}{R_2+R_3} & \dfrac{-R_3}{R_2+R_3} \\ \dfrac{R_3}{R_2+R_3} & \dfrac{R_1R_2 + R_1R_3 + R_2R_3}{R_2+R_3} \end{bmatrix}$$

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Ex:

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \longrightarrow \left[\begin{array}{l} \text{ABCD matrix} \\ \text{in terms of } r_{11}, r_{12}, r_{21}, r_{22} \end{array}\right]$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \begin{array}{l} V_1 = r_{11} I_1 + r_{12} I_2 \\ V_2 = r_{21} I_1 + r_{22} I_2 \end{array}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{r_{11}}{r_{21}}$$

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} \qquad V_2 = 0 = r_{21} I_1 + r_{22} I_2$$

$$I_1 = \frac{-r_{22} I_2}{r_{21}}$$

$$V_1 = r_{11} I_1 + r_{12} I_2 = \frac{-r_{11} r_{22} I_2}{r_{21}} + r_{12} I_2$$

$$V_1 = \left[\frac{-r_{11} r_{22}}{r_{21}} + r_{12}\right] I_2 \qquad B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = \frac{r_{11} r_{22}}{r_{21}} - r_{12}$$

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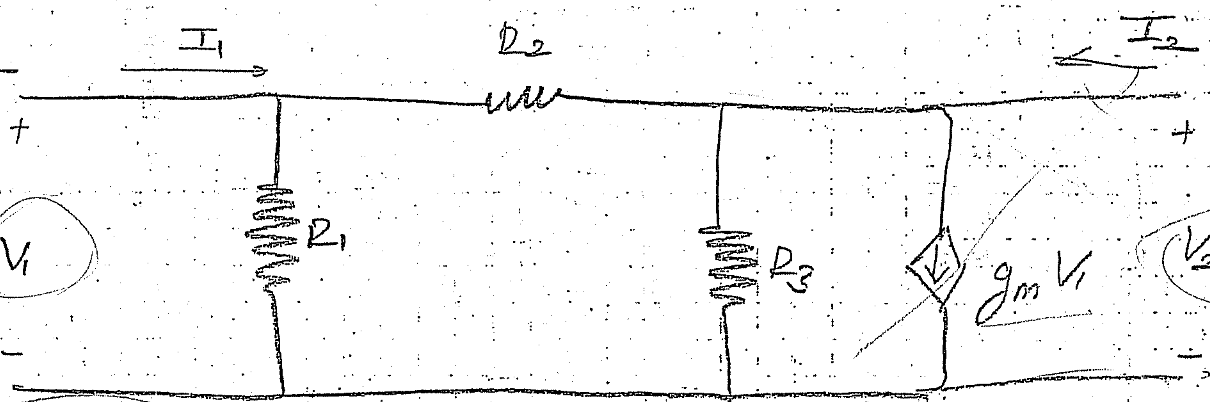
$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{r_{21}} \qquad D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{r_{22}}{r_{21}}$$

H.W (ABCD) $\rightarrow$ $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$

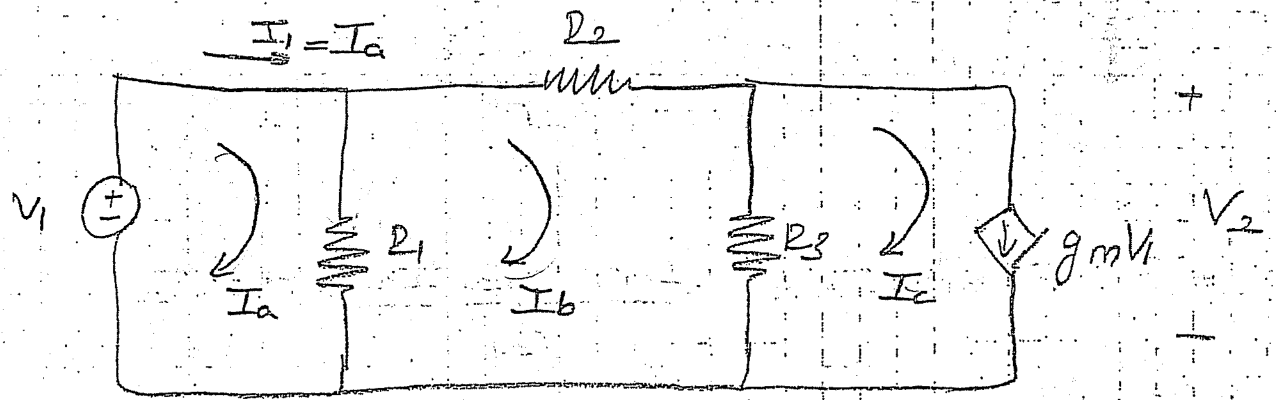
known

Ex:



Z matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad r_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$



(1) $-V_1 + R_1(I_a - I_b) = 0$

(2) $R_1(I_b - I_a) + R_2(I_b) + R_3(I_b - I_c) = 0$

(3) $I_c = g_m V_1$

(1) ⊕ (2) $\quad -V_1 + R_2 I_b + R_3 (I_b - I_c) = 0$ ← put (3)

$$-V_1 + R_2 I_b + R_3 \left[I_b - g_m V_1 \right] = 0$$

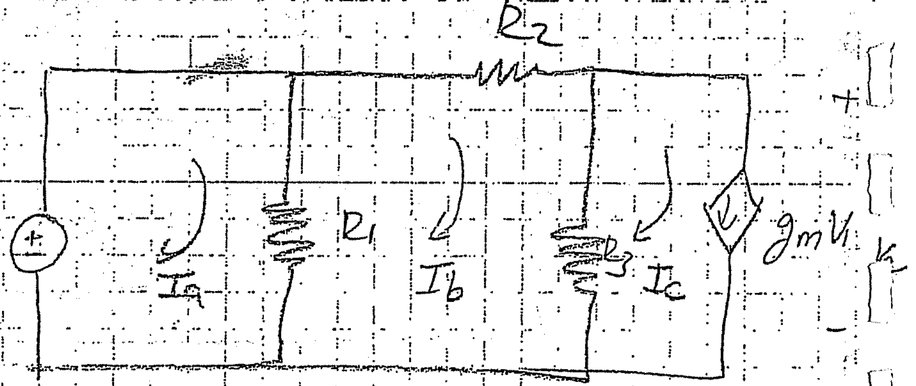
$$\boxed{I_b = \frac{1 + g_m R_3}{R_2 + R_3} V_1} \longrightarrow \text{put (1)}$$

$$-V_1 + R_1 I_a - R_1 \left(\frac{1 + g_m R_3}{R_2 + R_3} \right) V_1 = 0$$

$$R_1 I_a = V_1 \left[1 + \frac{R_1 + g_m R_1 R_3}{R_2 + R_3} \right]$$

$$\frac{V_1}{I_a} = \frac{R_1 (R_2 + R_3)}{R_2 R_3 + R_1 + g_m R_1 R_3} = \left. \frac{V_1}{I_1} \right|_{I_2 = 0} = r_{11}$$

$$r_{21} = \left. \frac{V_2}{I_1} \right|_{I_2 = 0}$$



$$V_2 = R_3 \left(I_b - I_c \right)$$

$$V_2 = R_3 \left(\frac{1 + g_m R_3}{R_1 + R_3} V_1 - g_m V_1 \right) = R_3 \left(\frac{1 - g_m R_2}{R_2 + R_3} \right) V_1$$

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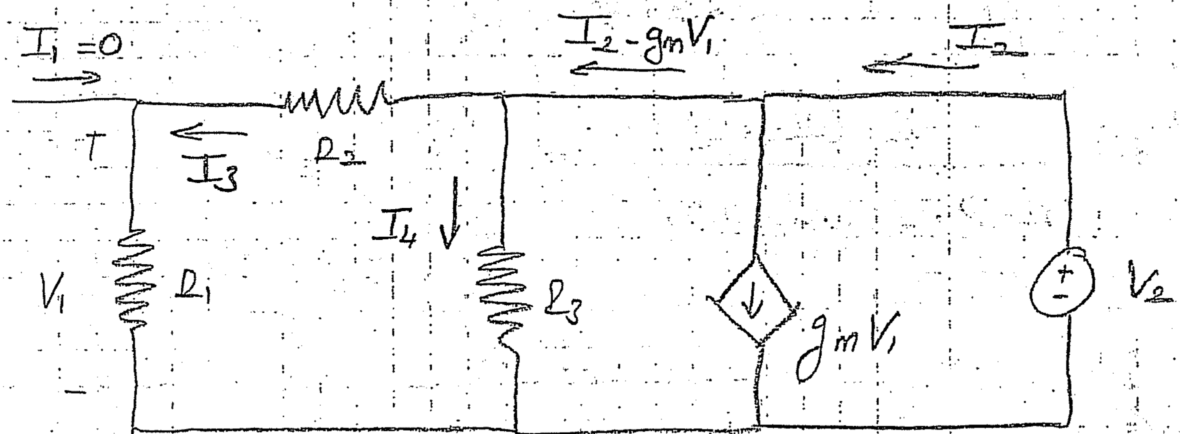
$$V_2 = \frac{R_3\,[1 - g_m R_2]}{R_2 + R_3}\, V_1$$

$$\frac{R_1 (R_2 + R_3)}{R_2 + R_3 + R_1 + g_m R_1 R_3}\, I_1$$

$$r_{21} = \frac{R_1 R_3\,[1 - g_m R_2]}{R_2 + R_3 + R_1 + g_m R_1 R_3} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

$$r_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} \qquad r_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$



$$V_1 = R_1 . I_3 \qquad I_3 = \underbrace{(I_2 - g_m V_1)}\, \frac{R_3}{R_1 + R_2 + R_3}$$

$$V_1 = \frac{R_1 R_3 (I_2 - g_m V_1)}{R_1 + R_2 + R_3} = \frac{R_1 R_3}{R_1 + R_2 + R_3} I_2 - \frac{g_m R_1 R_3 V_1}{R_1 + R_2 + R_3}$$

$$\left(1 + \frac{g_m R_1 R_3}{R_1 + R_2 + R_3}\right) V_1 = \frac{R_1 R_3}{R_1 + R_2 + R_3} I_2$$

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$$\left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{R_1+R_3}{R_1+R_2+R_3+g_m R_1 R_3} = r_{12}$$

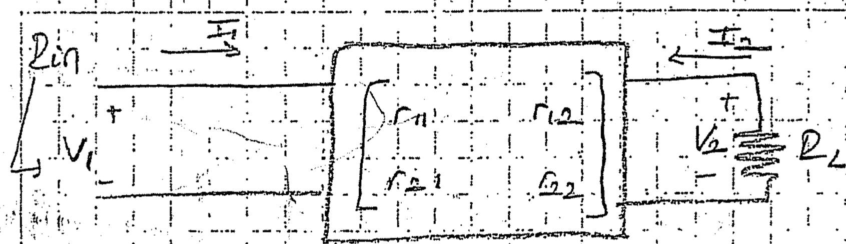
$$I_4 = (I_2 - g_m V_1)\frac{(R_1+R_2)}{R_1+R_2+R_3}$$

$$V_2 = I_4 R_3 = \left(I_2 - g_m \frac{R_1 R_3}{R_1+R_2+R_3+g_m R_1 R_3} I_2\right)\frac{(R_1+R_2) R_3}{R_1+R_2+R_3}$$

$$r_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \left[1 - g_m \frac{R_1 R_3}{R_1+R_2+R_3+g_m R_1 R_3}\right]\frac{(R_1+R_2) R_3}{R_1+R_2+R_3}$$

$$r_{22} = \frac{(R_1+R_2) R_3}{R_1+R_2+R_3+g_m R_1 R_3}$$

Terminated Two Ports:



$$R_{in} = \frac{V_1}{I_1}$$

$$\frac{V_2}{I_2} = -R_L$$

$$V_1 = r_{11} I_1 + r_{12} I_2$$

$$V_2 = r_{21} I_1 + r_{22} I_2$$

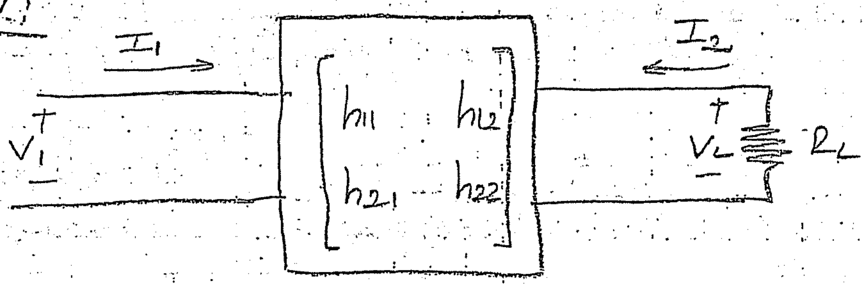
$$\frac{V_2}{I_2} = r_{21}\frac{I_1}{I_2} + r_{22} = -R_L$$

$$\frac{-R_L - r_{22}}{r_{21}} = \frac{I_1}{I_2} \qquad \frac{V_1}{I_1} = r_{11} + r_{12}\,\frac{I_2}{I_1}$$

$$\frac{V_1}{I_1} = r_{11} + r_{12}\,\frac{r_{21}}{-(R_L + r_{22})}$$

$$\frac{V_1}{I_1} = R_{in} = r_{11} - \frac{r_{12}\, r_{21}}{(R_L + r_{22})}$$

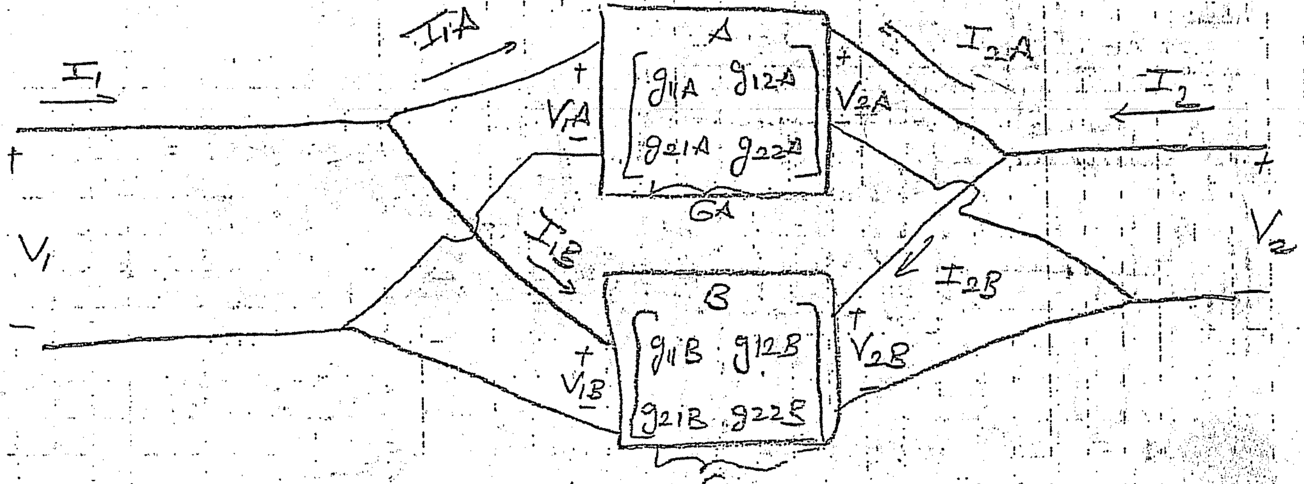
H.W.:



Find $R_{in} = \frac{V_1}{I_1}$ in terms of hybrid parameters.

Interconnection of Two Ports:

Parallel Connection:



$I_1 = I_{1A} + I_{1B}$ $\qquad V_{1A} = V_{1B} = V_1$

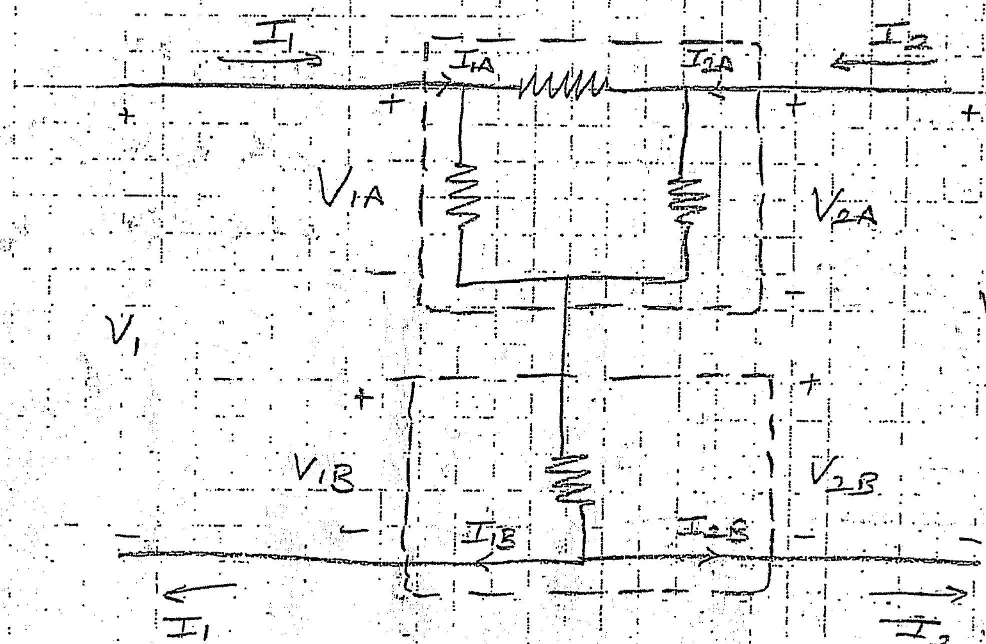
$I_2 = I_{2A} + I_{2B}$ $\qquad V_{2A} = V_{2B} = V_2$

$$\begin{bmatrix} I_{1A} \\ I_{2A} \end{bmatrix} = G_A \begin{bmatrix} V_{1A} \\ V_{2A} \end{bmatrix} \qquad \begin{bmatrix} I_{1B} \\ I_{2B} \end{bmatrix} = G_B \begin{bmatrix} V_{1B} \\ V_{2B} \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_{1A} + I_{1B} \\ I_{2A} + I_{2B} \end{bmatrix} = G_A \begin{bmatrix} \overset{V_1}{V_{1A}} \\ \underset{V_2}{V_{2A}} \end{bmatrix} + G_B \begin{bmatrix} \overset{V_1}{V_{1B}} \\ \underset{V_2}{V_{2B}} \end{bmatrix} = (G_A + G_B) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$G = \begin{bmatrix} g_{11}A + g_{11}B & g_{12}A + g_{12}B \\ g_{21}A + g_{21}B & g_{22}A + g_{22}B \end{bmatrix}$$

Series connection :



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$I_1 = I_{1A} = I_{1B}$ $\qquad V_1 = V_{1A} + V_{1B}$

$I_2 = I_{2A} = I_{2B}$ $\qquad V_2 = V_{2A} + V_{2B}$

$$\begin{bmatrix} V_{1A} \\ V_{2A} \end{bmatrix} = R_A \begin{bmatrix} I_{1A} \\ I_{2A} \end{bmatrix}$$

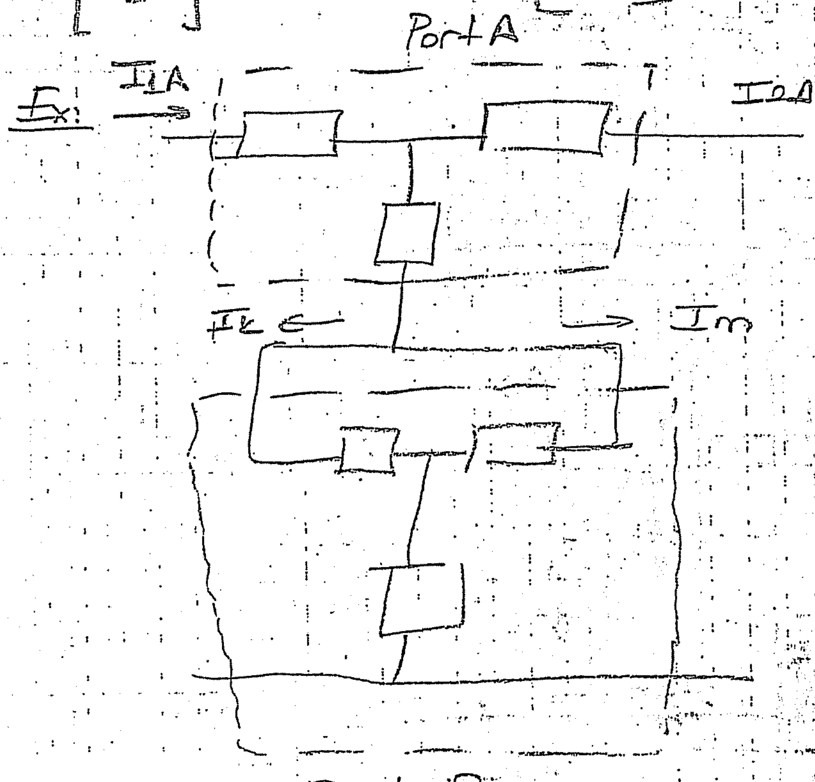
I_1, I_2

$$\begin{bmatrix} V_{1B} \\ V_{2B} \end{bmatrix} = R_B \begin{bmatrix} I_{1B} \\ I_{2B} \end{bmatrix}$$

I_1, I_2

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \left(R_A + R_B \right) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Ex:

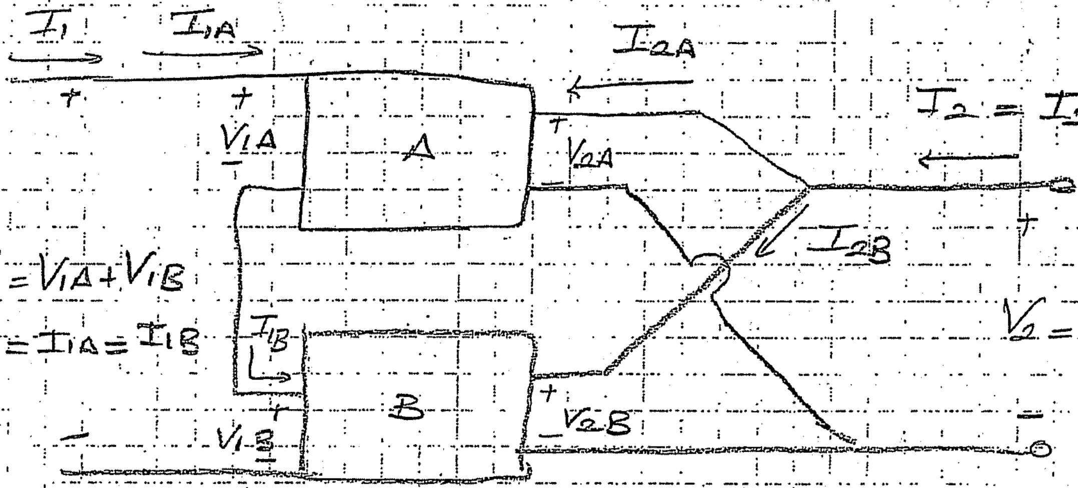


$I_k \overset{?}{=} I_{1A}$

$I_m = I_{2A}$

not a series connection.

Series _ Parallel Connection.



$V_1 = V_{1A} + V_{1B}$

$I_1 = I_{1A} = I_{1B}$

$I_2 = I_{2A} + I_{2B}$

$V_2 = V_{2A} = V_{2B}$

$$* \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$2 \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$V_1 = V_{1A} + V_{1B}$

$I_2 = I_{2A} + I_{2B}$

$$\begin{bmatrix} V_{1A} \\ I_{2A} \end{bmatrix} = \begin{bmatrix} h_{11A} & h_{12A} \\ h_{21A} & h_{22A} \end{bmatrix} \begin{bmatrix} I_{1A} \to I_1 \\ V_{2A} \to V_2 \end{bmatrix}$$

$$\begin{bmatrix} V_{1B} \\ I_{2B} \end{bmatrix} = \begin{bmatrix} h_{11B} & h_{12B} \\ h_{21B} & h_{22B} \end{bmatrix} \begin{bmatrix} I_{1B} \to I_1 \\ V_{2B} \to V_2 \end{bmatrix}$$

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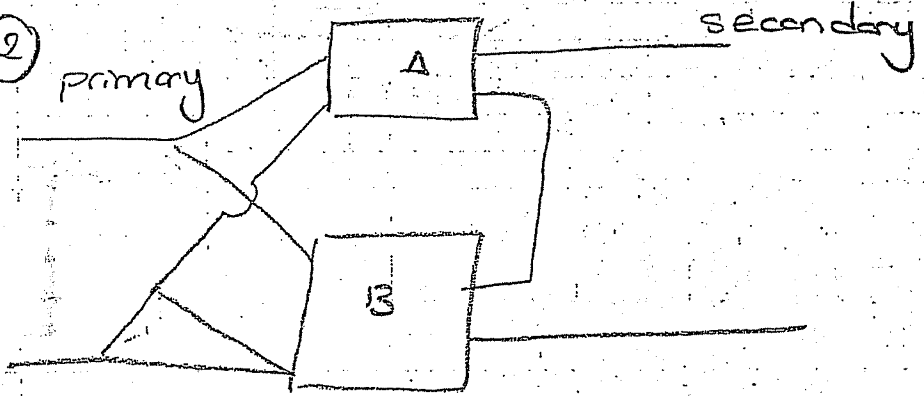
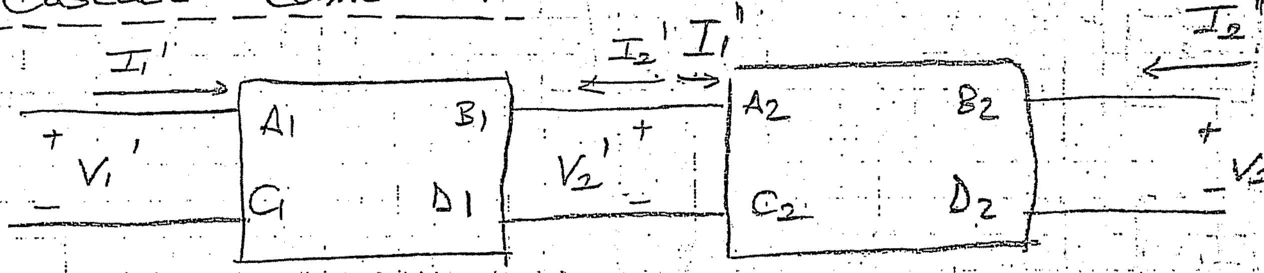
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11A} + h_{11B} & h_{12A} + h_{12B} \\ h_{21A} + h_{21B} & h_{22A} + h_{22B} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

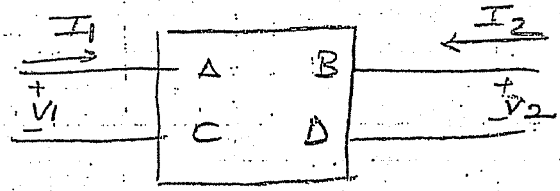
(2)



Cascade Connection:



$$|||$$



$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2' \rightarrow V_1'' \\ -I_2' \rightarrow I_1'' \end{bmatrix} \qquad \begin{bmatrix} V_1'' \\ I_1'' \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2'' \\ -I_2'' \end{bmatrix}$$

$V_2' = V_1''$

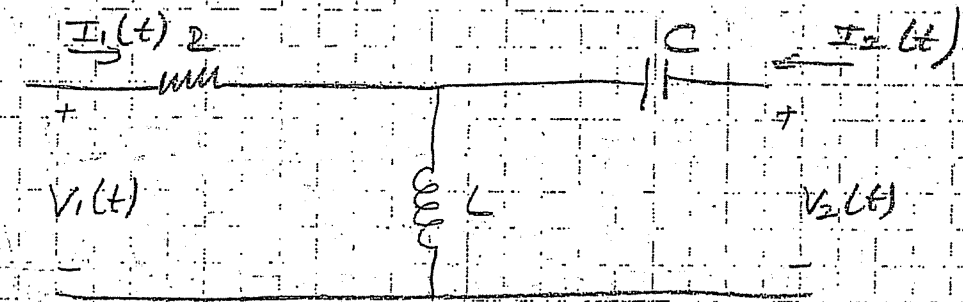
$I_2' = -I_1''$

$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}}_{ABCD}\begin{bmatrix} V_2'' \\ -I_2'' \end{bmatrix}$$

$V_1' \rightarrow V_1$, $I_1' \rightarrow I_1$, $V_2'' \rightarrow V_2$, $-I_2'' \rightarrow -I_2$

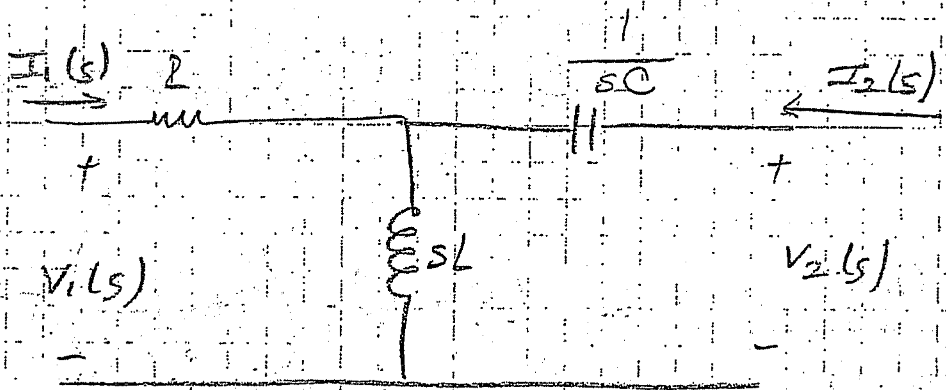
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Ex:



Find port equation in laplace domain ABCD

$\mathcal{L}$



$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_2(s) \\ -I_2(s) \end{bmatrix}$$

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$$V_1(s) = R I_1(s) + sL\left(I_1(s) + I_2(s)\right) = \left[R + sL\right] I_1(s) + sL\, I_2(s)$$

$$V_2(s) = \frac{1}{sC} I_2(s) + sL\left(I_1(s) + I_2(s)\right) = sL\, I_1(s) + \left(sL + \frac{1}{sC}\right] I_2(s)$$

$$A = \left.\frac{V_1(s)}{V_2(s)}\right|_{-I_2(s)=0} = \frac{R + sL}{sL}$$

$$B = \left.\frac{V_1(s)}{-I_2(s)}\right|_{V_2(s)=0}$$

$$V_2(s) = 0 \Rightarrow sL\, I_1(s) + \left[sL + \frac{1}{sC}\right] I_2(s) = 0$$

$$I_1(s) = \frac{-\left(sL + \frac{1}{sC}\right) I_2(s)}{sL} = -\left(1 + \frac{1}{s^2 LC}\right) I_2(s)$$

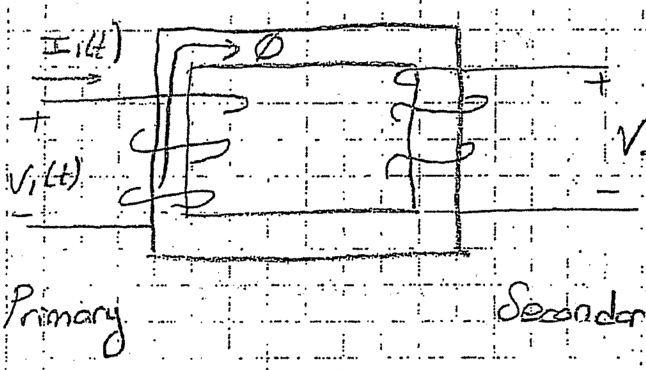
$$V_1(s) = -(R + sL)\left(1 + \frac{1}{s^2 LC}\right) I_2(s) + sL\, I_2(s)$$

$$B = \left.\frac{V_1(s)}{-I_2(s)}\right|_{V_2(s)=0} = (R + sL)\left(1 + \frac{1}{s^2 LC}\right) - sL$$

$$C = \left. \frac{I_1(s)}{V_2(s)} \right|_{I_2(s)=0} = \frac{1}{sL}$$

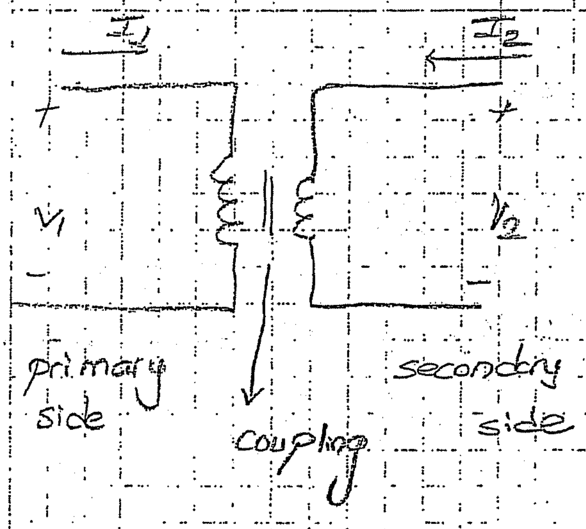
$$D = \left. \frac{I_1(s)}{-I_2(s)} \right|_{V_2(s)=0} = \frac{sL + \frac{1}{sC}}{sL} = 1 + \frac{1}{s^2 LC}$$

Coupled Inductors and Transformers:



$V_2(t) \neq 0$ $\qquad \frac{d\phi}{dt} = V$

Mutual inductance: The coupling (relation) between the current in the first coil and produced voltage due to this current in the second coil.



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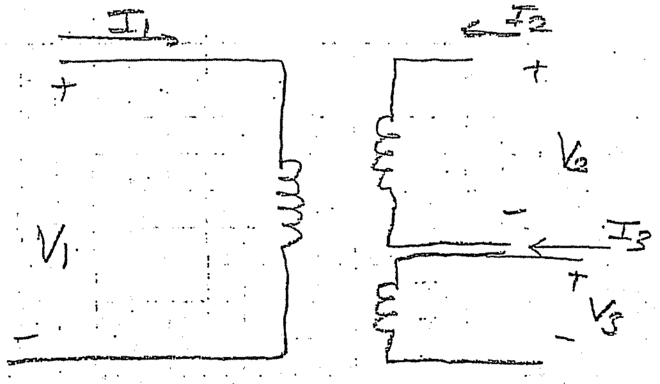
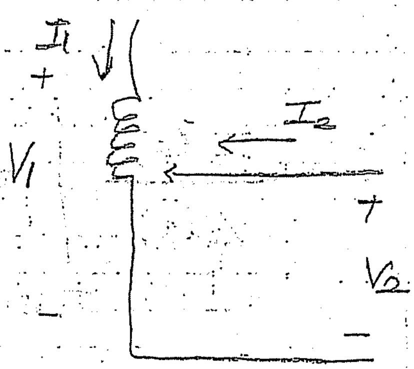
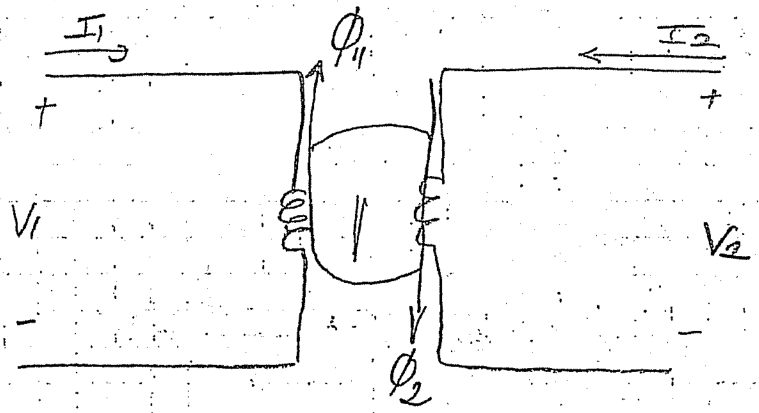




LTI 2-branch coupled inductors



$$\phi_1 = L_1 I_1 + M I_2 \qquad \phi_2 = M I_1 + L_2 I_2$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

L_1 = Self inductance of primary coil

L_2 = " " " secondary "

M = mutual inductance

by Faraday's Law $\quad V_k = \dfrac{d\phi_k}{dt} \qquad k = 1, 2, \ldots, n$

$$V_1(t) = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

$$\underline{V}_2(t) = M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt}$$

$$\underline{V} = L \frac{d}{dt} \underline{I}$$

$$L = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$$

If $\det[L] \neq 0$; L is non-singular

$\Gamma \triangleq L^{-1}$; reciprocal inductance matrix

$$\underline{V}\, dt = L\, \underline{dI}$$

$$\int_{t_0}^{t} \underline{V}(\tau)\, d\tau = L \int_{I_0}^{I(t)} \underline{dI}$$

$$\int_{t_0}^{t} \underline{V}(\tau)\, d\tau = L \left[\underline{I}(t) - \underline{I}_0 \right]$$

$$L^{-1} \int_{t_0}^{t} \underline{V}(\tau) d\tau = \left[\underline{I}(t) - \underline{I}_0 \right]$$

$$\boxed{\underline{I}(t) = \underline{I}_0 + \Gamma \int_{t_0}^{t} \underline{V}(\tau)\, d\tau}$$

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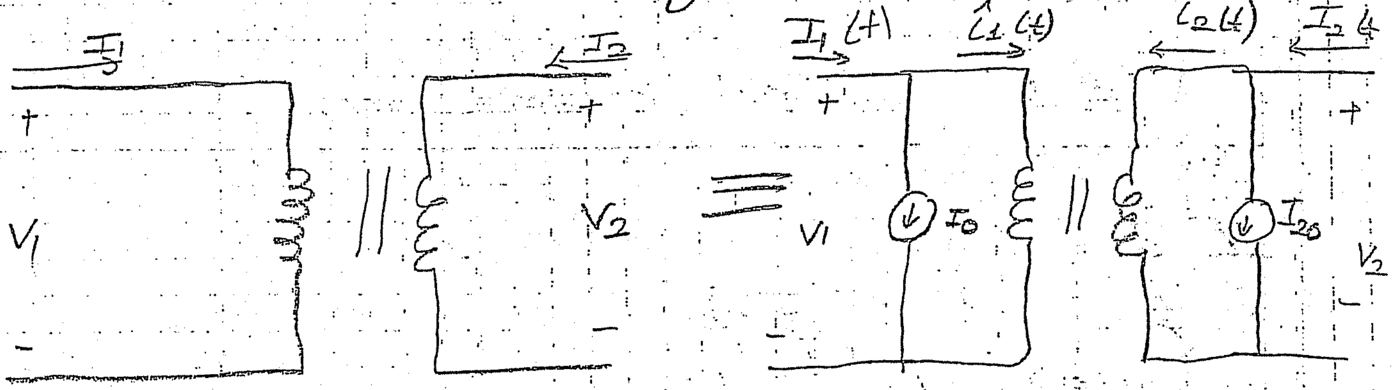
$$L = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \longrightarrow \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12} & \Gamma_{22} \end{bmatrix}$$

$$i_1(t) = i_1(t_0) + \Gamma_{11} \int_{t_0}^{t} V_1(\tau)\,d\tau + \Gamma_{12} \int_{t_0}^{t} V_2(\tau)\,d\tau$$

$$i_2(t) = i_2(t_0) + \Gamma_{12} \int_{t_0}^{t} V_1(\tau)\,d\tau + \Gamma_{22} \int_{t_0}^{t} V_2(\tau)\,d\tau$$

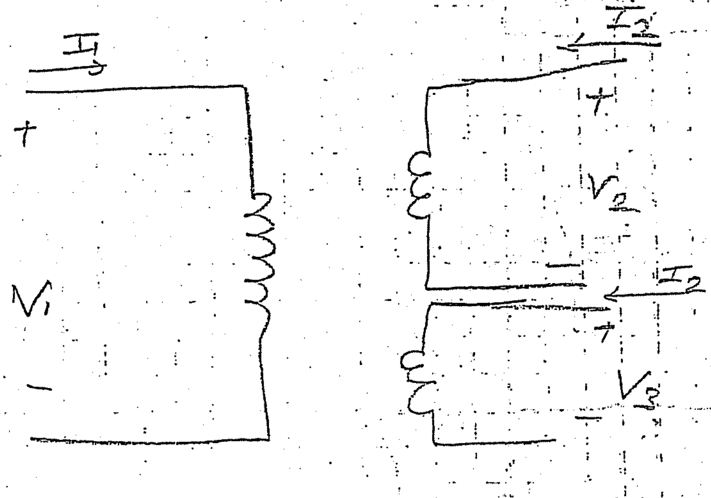
initial currents over primary and secondary coils.



$\hat{i}_1(t_0) = 0 \qquad \hat{i}_2(t_0) = 0$

$i_1(t_0) = I_{10}$

$i_2(t_0) = I_{20}$



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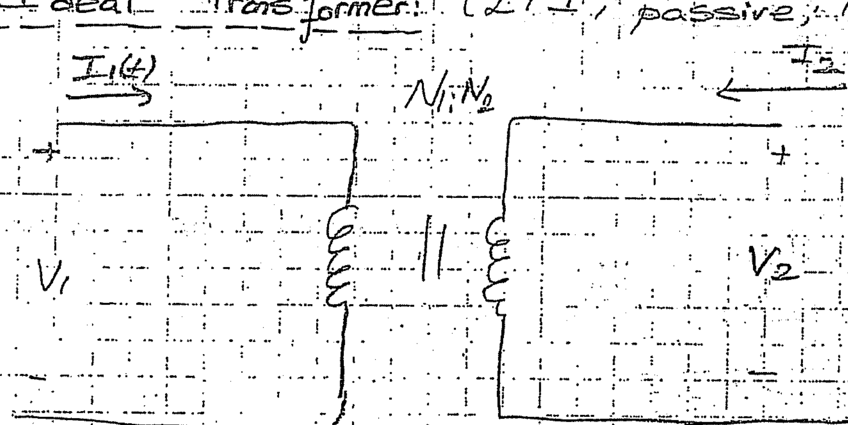
$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{12} & L_2 & M_{23} \\ M_{13} & M_{23} & L_3 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Ideal Transformer: (LTI, passive, lossless circuit element that is used for changing direction and magnitude of AC current and voltage.)



N_1 = Number of turns of windings in primary
N_2 = " " " " " " secondary

$n \triangleq \frac{N_1}{N_2}$ ratio of turns

$V_1(t) = n V_2(t) \qquad i_2(t) = -n\, i_1(t)$

Instantaneous Power $= P(t) = \overbrace{V_1(t) I_1(t)}^{P_1(t)} + \overbrace{V_2(t) I_2(t)}^{P_2(t)}$

$$= n V_2(t) \left(\frac{-I_2(t)}{n} \right) + V_2(t) I_2(t)$$

$P(t) = -V_2(t) I_2(t) + V_2(t) I_2(t) = 0$

$\Rightarrow$ No power dissipation $\Rightarrow$ no loss

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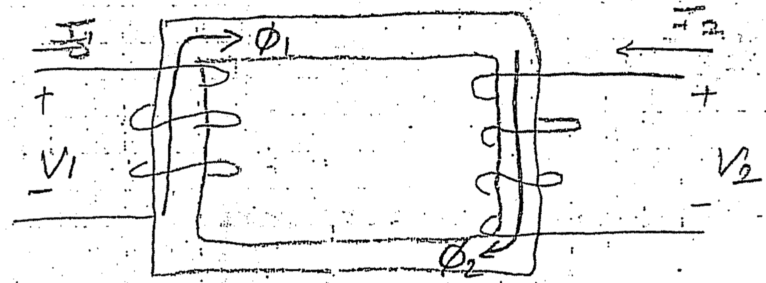
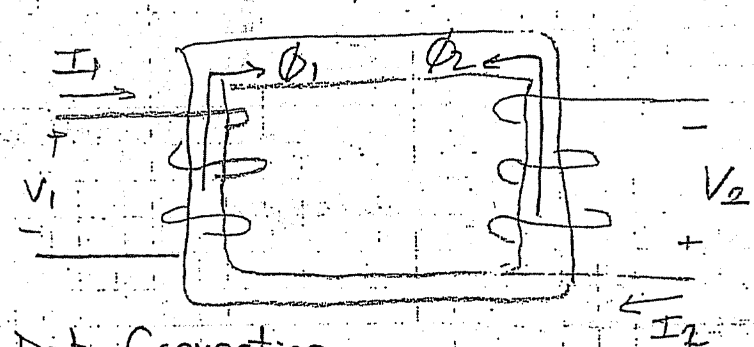
For ideal transformers (L_1, L_2) (self inductance values) of inductors are assumed to be equal to ∞.
(Observed from the equations $V_1 = n V_2$ $\quad I_2 = -n I_1$)



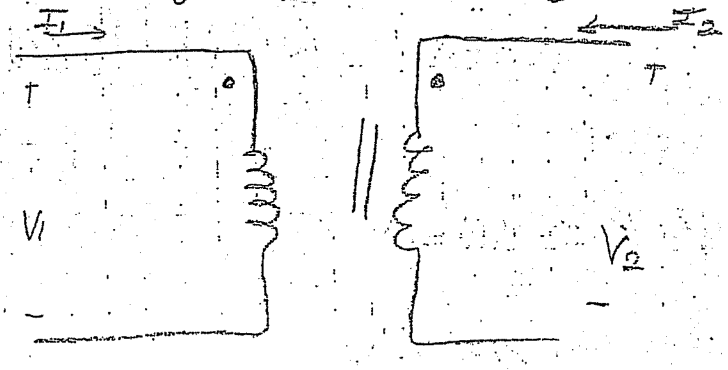
ϕ_1 and ϕ_2 are additive



ϕ_1 and ϕ_2 are subtructive

Dot Convention
If positive rate of change of current of the primary part produces (+)'ve effect on the secondary side induced voltage or vice versa, the mutual inductance values (M) will be (+)'ve; otherwise mutual inductance values (M) will be (−)'ve.

Actually the sign of M depends on
1. The spatial orientation of coil.
2. The reference marks given to the coil voltages and currents.



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

$M > 0$ additive

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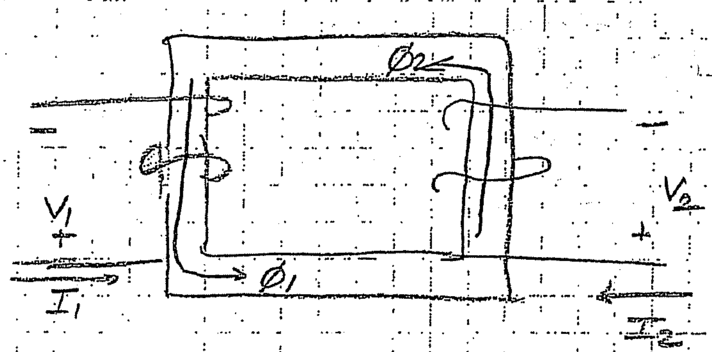
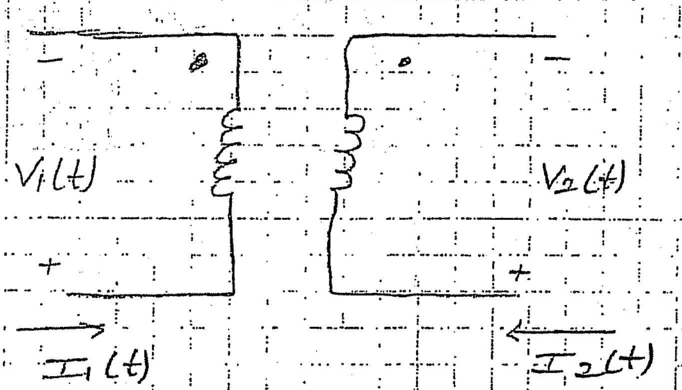
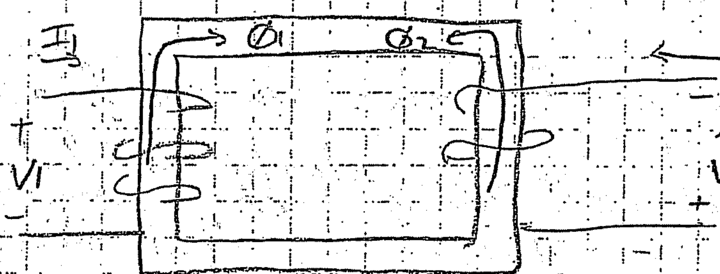




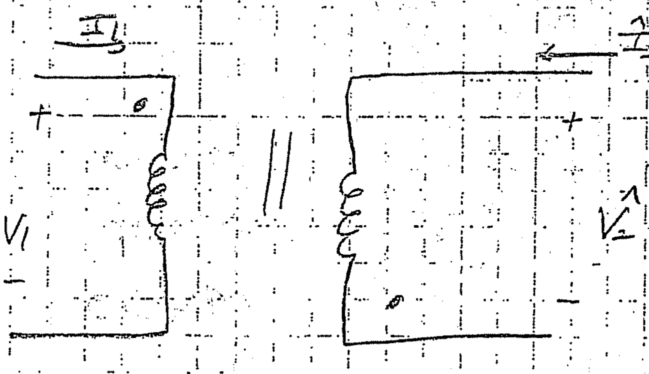
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

$M > 0$ additive



$\hat{I}_2 = -I_2$

$\hat{V}_2 = -V_2$



$$\begin{bmatrix} V_1 \\ \hat{V}_2 \end{bmatrix} = \begin{bmatrix} L_1 & -M \\ -M & L_2 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{d\hat{I}_2}{dt} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

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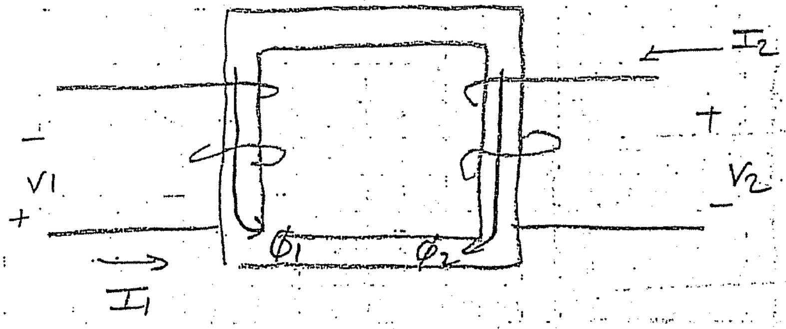
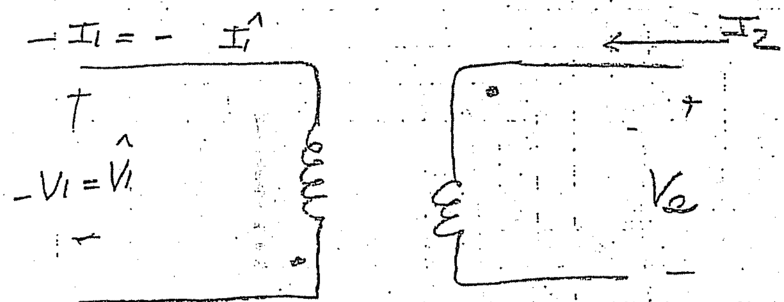
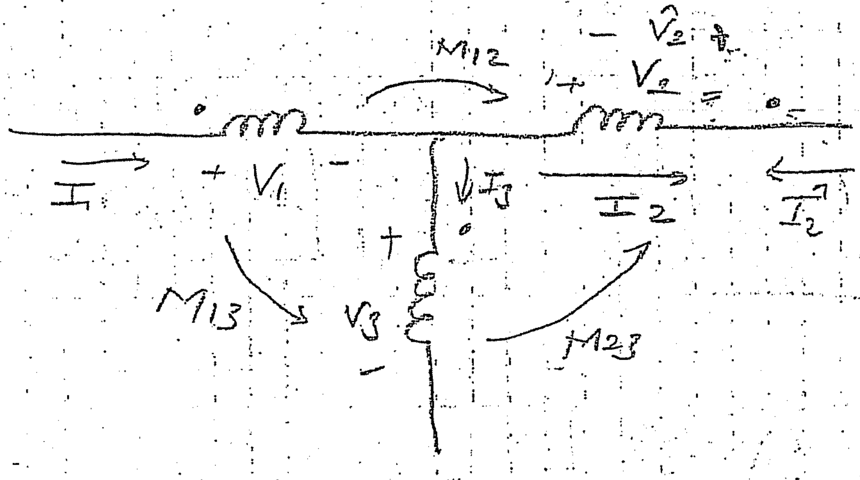




$$\begin{bmatrix} \hat{V_1} \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & -M \\ -M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d\hat{I_1}}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

* In a circuit: If the current enters the coil from the "•" direction and if the voltage polarity in "•" direction is chosen as positive polarity then take mutual inductance value to be '+'



$M_{12} > 0$

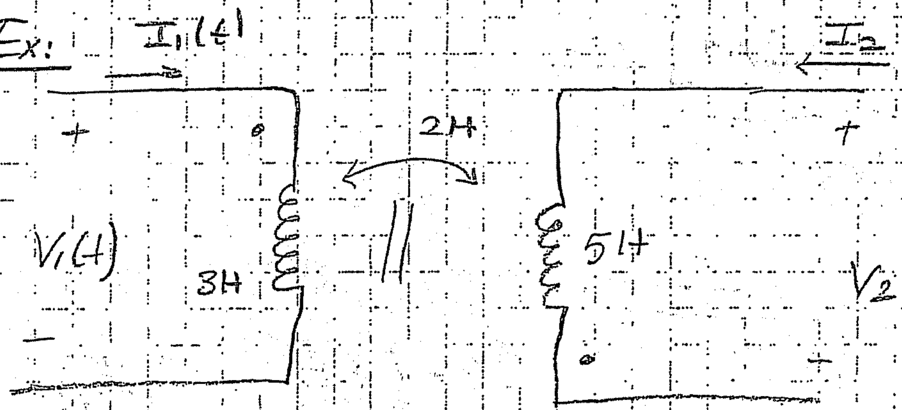
$M_{13} > 0$

$M_{23} > 0$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} L_1 & -M_{12} & M_{13} \\ -M_{12} & L_2 & M_{13} \\ M_{13} & -M_{23} & L_3 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \\ \frac{dI_3}{dt} \end{bmatrix}$$

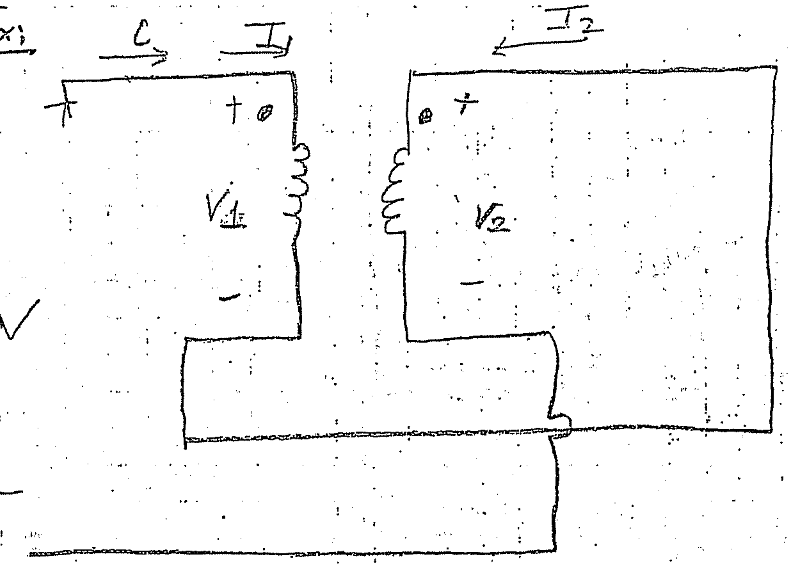
$$\begin{bmatrix} V_1 \\ \hat{V}_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{12} & L_2 & M_{13} \\ M_{13} & M_{23} & L_3 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{d\hat{I}_2}{dt} \\ \frac{dI_3}{dt} \end{bmatrix}$$

Ex:



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

Ex:



$$V_1 = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

$$V_2 = M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt}$$

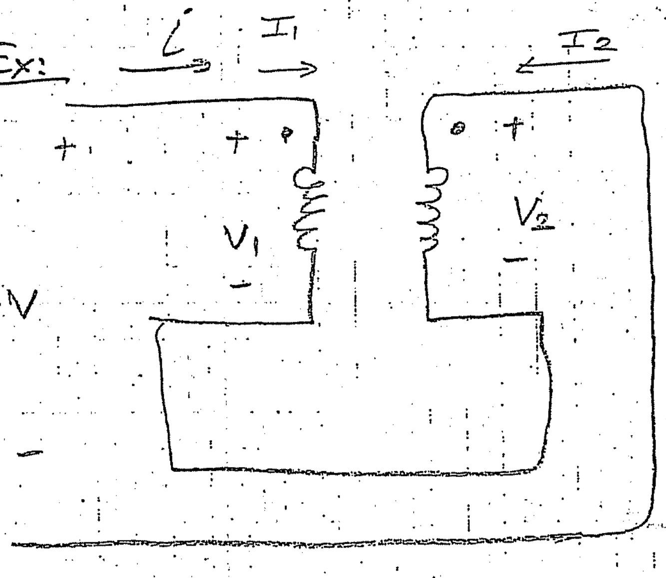
$$i = I_1(t) = I_2(t)$$

$$V = V_1(t) + V_2(t) = (L_1 + M)\frac{dI_1}{dt} + (M + L_2)\frac{dI_2}{dt}$$

$$= (L_1 + M)\frac{di}{dt} + (M + L_2)\frac{di}{dt}$$

$$V(t) = \underbrace{[L_1 + L_2 + 2M]}_{Leq} \frac{di}{dt}$$

Ex:



$$V_1 = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

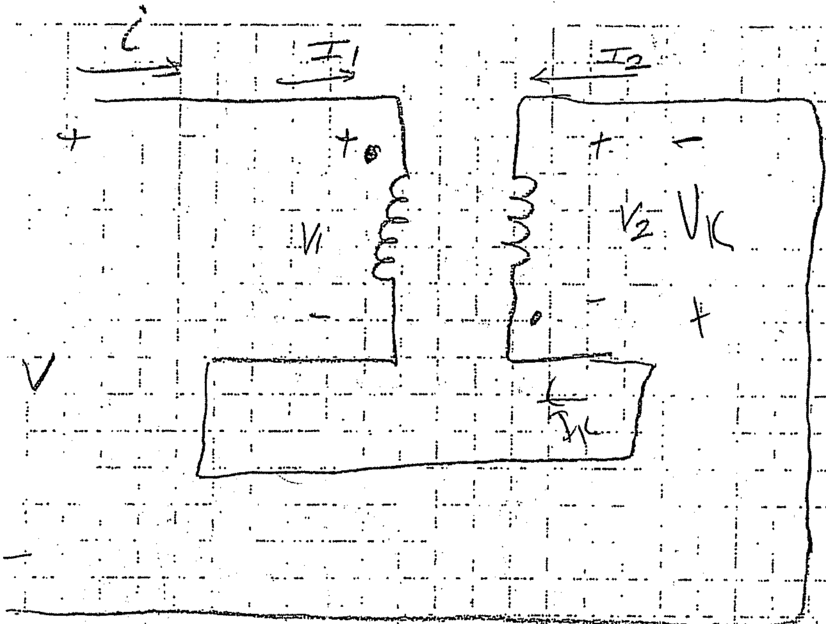
$$V_2 = L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}$$

$$i = I_1(t) = -I_2(t)$$

$$V = V_1(t) - V_2(t) = (L_1 - M)\frac{dI_1}{dt} + (M - L_2)\frac{dI_2}{dt}$$

$$= (L_1 - M)\frac{di}{dt} + (M - L_2)\frac{d[-i]}{dt}$$

$$V = [L_1 + L_2 - 2M]\frac{di}{dt}$$



$$V_1 = L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt}$$

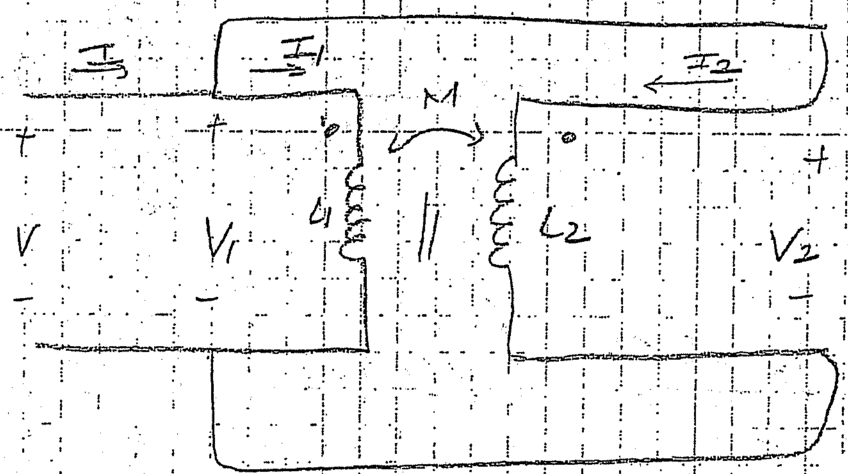
$$V_2 = -M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt}$$

$$i = I_1(t) = -I_2(t)$$

$$V(t) = V_1(t) - V_2(t) = [L_1+M] \frac{dI_1}{dt} - [L_2+M] \frac{dI_2}{dt}$$

$$= (L_1+M) \frac{di}{dt} - [L_2+M] \frac{d}{dt}(-i)$$

$$V = \underbrace{(L_1+L_2+2M)}_{Leq} \frac{di}{dt}$$



$$L_1 L_2 \neq M^2$$

$$V = V_1 = V_2$$

$$I = I_1 + I_2$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

$$V = M \frac{dI}{dt}$$

$$I = I_0 + \Gamma \int_{t_0}^{t} V(\tau)\, d\tau$$

$$L = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \longrightarrow L^{-1} = \frac{\begin{bmatrix} L_2 & -M \\ -M & L_1 \end{bmatrix}}{L_1 L_2 - M^2}$$

$$I = \underbrace{\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix}}_{\text{currents}} = \underbrace{\begin{bmatrix} I_{10} \\ I_{20} \end{bmatrix}}_{\text{initial conditions}} + \frac{\begin{bmatrix} L_2 & -M \\ -M & L_1 \end{bmatrix}}{L_1 L_2 - M^2} \int_{t_0}^{t} \begin{bmatrix} V_1(\tau) \\ V_2(\tau) \end{bmatrix} d\tau$$

$$I_1(t) = I_{10} + \frac{L_2}{L_1 L_2 - M^2} \int_{t_0}^{t} V_1(\tau)\, d\tau + \frac{-M}{L_1 L_2 - M^2} \int_{t_0}^{t} V_2(\tau)\, d\tau$$

$$I_2(t) = I_{20} - \frac{M}{L_1 L_2 - M^2} \int_{t_0}^{t} V_1(\tau)\, d\tau + \frac{L_1}{L_1 L_2 - M^2} \int_{t_0}^{t} V_2(\tau)\, d\tau$$

$$I = \underbrace{I_{10} + I_{20}}_{I_0} + \underbrace{\frac{L_1 + L_2 - 2M}{L_1 L_2 - M^2}}_{\mathcal{L}_{eq}^{-1} = \frac{1}{L_{eq}}} \int_{t_0}^{t} V(\tau)\, d\tau$$

$$\mathcal{L}_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

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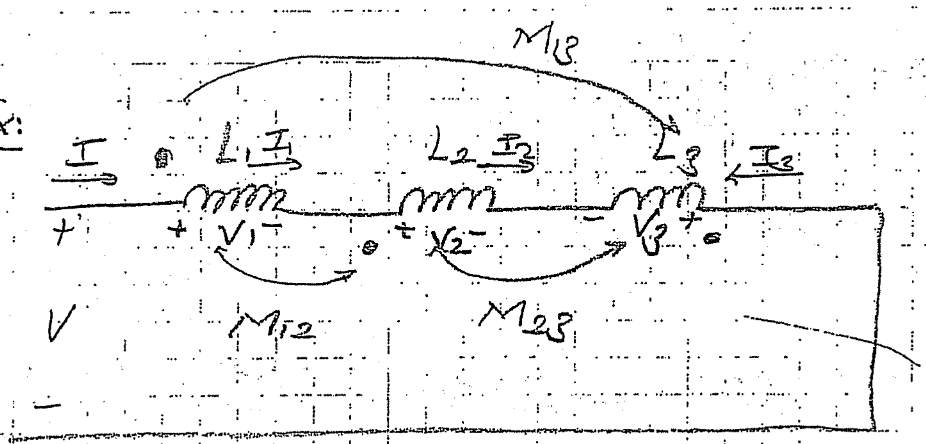
Ex:



$$L_{eq}\frac{dI}{dt}=V$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{12} & L_2 & M_{23} \\ M_{13} & M_{23} & L_3 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \\ \frac{dI_3}{dt} \end{bmatrix}$$

$$V = V_1 + V_2 - V_3$$

$$= L_1\frac{dI_1}{dt} + M_{12}\frac{dI_2}{dt} + M_{13}\frac{dI_3}{dt}$$

$$+ M_{12}\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} + M_{23}\frac{dI_3}{dt}$$

$$- M_{13}\frac{dI_1}{dt} - M_{23}\frac{dI_2}{dt} - L_3\frac{dI_3}{dt}$$

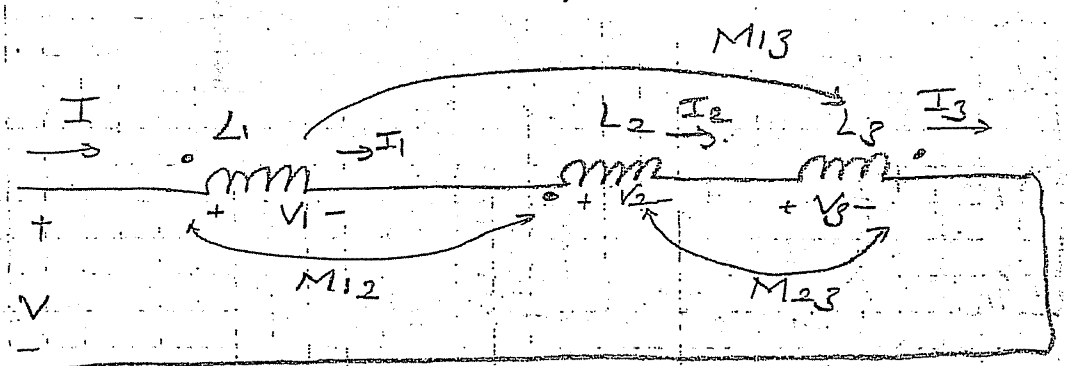
$$V = \left(L_1 + M_{12} - M_{13}\right)\frac{dI_1}{dt} + \left(M_{12} + L_2 - M_{23}\right)\frac{dI_2}{dt}$$

$$+ \left(M_{13} + M_{23} - L_3\right)\frac{dI_3}{dt}$$

$I_1 = I$ $\quad I_2 = I$ $\quad I_3 = -I$

$$V = (L_1 + M_{12} - M_{13})\frac{dI}{dt} + (M_{12} + L_2 - M_{23})\frac{dI}{dt} + (L_3 - M_{13} - M_{23})\frac{dI}{dt}$$

$$V = \underbrace{\left(L_1 + L_2 + L_3 + 2M_{12} - 2M_{13} - 2M_{23}\right)}_{L_{eq}}\frac{dI}{dt}$$



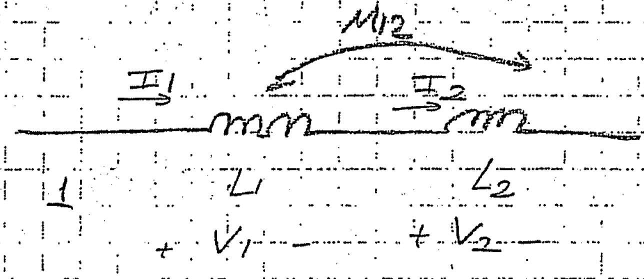
$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} L_1 & M_{12} & -M_{13} \\ M_{12} & L_2 & -M_{23} \\ -M_{13} & -M_{23} & L_3 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \\ \frac{dI_3}{dt} \end{bmatrix}$$

$V = V_1 + V_2 + V_3 \qquad I = I_1 = I_2 = I_3$

$$V = L_1\frac{dI_1}{dt} + M_{12}\frac{dI_2}{dt} - M_{13}\frac{dI_3}{dt} + M_{12}\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} - M_{23}\frac{dI_3}{dt} - M_{13}\frac{dI_1}{dt} - M_{23}\frac{dI_2}{dt} + L_3\frac{dI_3}{dt}$$

$$V = \underbrace{\left[L_1 + L_2 + L_3 + 2M_{12} - 2M_{13} - 2M_{23} \right]}_{L_{eq}} \frac{dI}{dt}$$

<u>Power and Energy</u>



Power absorbed by coil 1

$$P_1(t) = V_1(t)\, I_1(t)$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M_{12} \\ M_{12} & L_2 \end{bmatrix} \begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix}$$

$$P_1(t) = \left(L_1 \frac{dI_1}{dt} + M_{12} \frac{dI_2}{dt} \right) I_1$$

Power absorbed by coil 2

$$P_2(t) = V_2(t)\, I_2(t) = \left(M_{12} \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} \right) I_2$$

Total power $= P_1 + P_2$

$$P_1(t) + P_2(t) = P(t) = L_1 I_1 \frac{dI_1}{dt} + L_2 I_2 \frac{dI_2}{dt} + M_{12} \left(\frac{dI_2}{dt} I_1 + \frac{dI_1}{dt} I_2 \right)$$

$$= \frac{d}{dt}\left[\frac{1}{2} L_1 I_1^2(t) \right] + \frac{d}{dt}\left[\frac{1}{2} L_2 I_2^2(t) \right]$$

$$+ M_{12} \frac{d}{dt} \left[I_1(t)\, I_2(t) \right]$$

$$P(t) = \frac{d}{dt}\left[\frac{1}{2} L_1 I_1^2(t) + \frac{1}{2} L_2 I_2^2(t) \mp M_{12} I_1(t) I_2(t)\right]$$

$W(t; t_0)$ is the stored energy in the coils between time instances starting from t_0 up to t.

$$W(t; t_0) = \int_{t_0}^{t} P(t')dt' = \int_{t_0}^{t} \frac{d}{dt'}\left[\frac{1}{2} L_1 I_1^2(t') + \frac{1}{2} L_2 I_2^2(t') \mp M_{12} I_1(t') I_2(t')\right] dt'$$

$$W(t; t_0) = \frac{1}{2} L_1 I_1^2(t) + \frac{1}{2} L_2 I_2^2(t) \mp M_{12} I_1(t) I_2(t) - \left[\frac{1}{2} L_1 I_{10}^2 + \frac{1}{2} L_2 I_{20}^2 \pm M_{12} I_{10} I_{20}\right]$$

$$\left(I_1(t_0) = I_{10} \quad I_2(t_0) = I_{20}\right)$$

Now assume $I_{10} = 0$ $I_{20} = 0$ (the inductors are initially uncharged)

$$W(t; t_0) = \frac{1}{2} L_1 I_1^2(t) + \frac{1}{2} L_2 I_2^2(t) \mp M_{12} I_1(t) I_2(t)$$

$$W(t; t_0) = \frac{1}{2} \underbrace{\begin{bmatrix} I_1(t) & I_2(t) \end{bmatrix}}_{} \underbrace{\begin{bmatrix} L_1 & M_{12} \\ M_{12} & L_2 \end{bmatrix}}_{\underline{L}} \underbrace{\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix}}_{\underline{I}(t)}$$

$$W(t; t_0) = \frac{1}{2} \underline{I}^T(t)$$

* The coupled inductor is passive element
$\Longleftrightarrow W(t) \geq 0$

$\Rightarrow \underset{\sim}{L}$ should be positive semi-definite.

$$\begin{bmatrix} L_1 & M_{12} \\ M_{12} & L_2 \end{bmatrix} \qquad \begin{array}{l} L_1 \geq 0 \\ L_2 \geq 0 \\ \det = L_1 L_2 - M_{12}^2 \geq 0 \qquad L_1 L_2 \geq M_{12}^2 \end{array}$$

<u>Coupling coefficient</u> $k = \dfrac{M_{12}}{\sqrt{L_1 L_2}} \leq 1$

if $M_{12} = 0 \Rightarrow k = 0$ (the coupling between the coils is zero)
if $k = 1$ (perfect coupling) $M_{12}^2 = L_1 L_2$ $\det = L_1 L_2 - M_{12}^2 = 0$

$\Rightarrow L$ matrix $= \begin{bmatrix} L_1 & M_{12} \\ M_{12} & L_2 \end{bmatrix}$ is singular

$\det[L] = 0$

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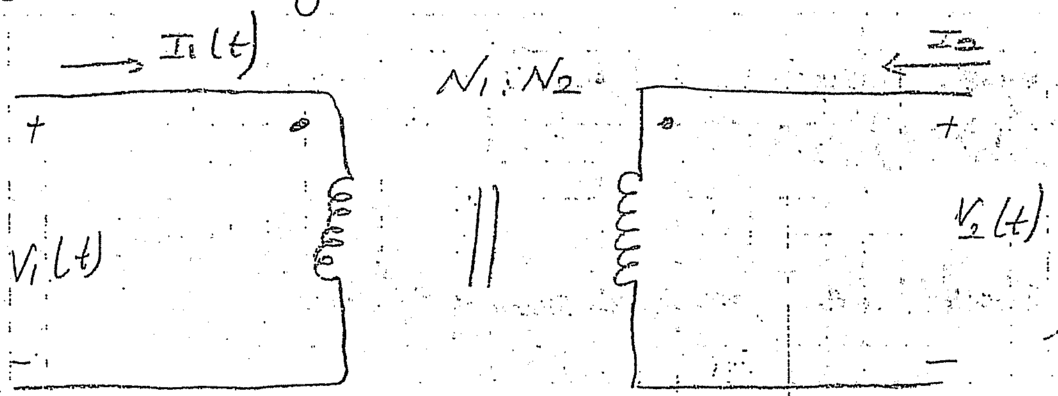
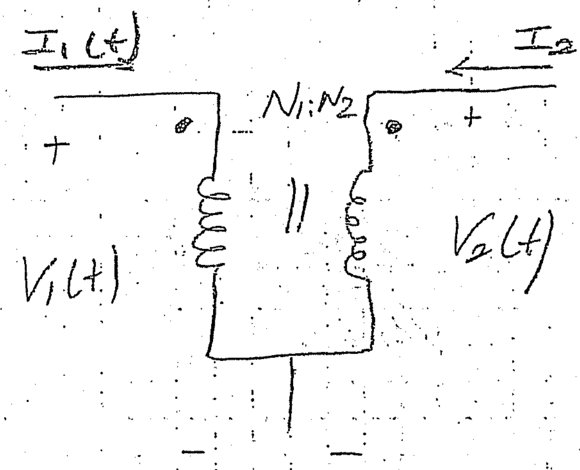
Ideal Transformers:



4 terminal, 2 branches



$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = n = \text{turn ratio}$$

N_1: # of primary turns

N_2: # of secondary turns

$$I_1 N_1 + I_2 N_2 = 0 \qquad \frac{I_1}{I_2} = \frac{-N_2}{N_1}$$

Ideal transformers

① No losses (no core loss; no winding loss, leakage loss)

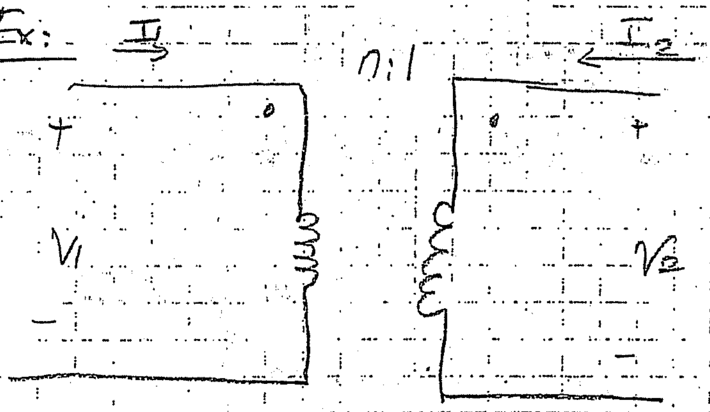
② $\mu \to \infty$ (the magnetic element core ~~should~~ is made of confine the total flux in the core only

③ $k = 1$ (perfect coupling between the coupled inductors)

L1 — 1

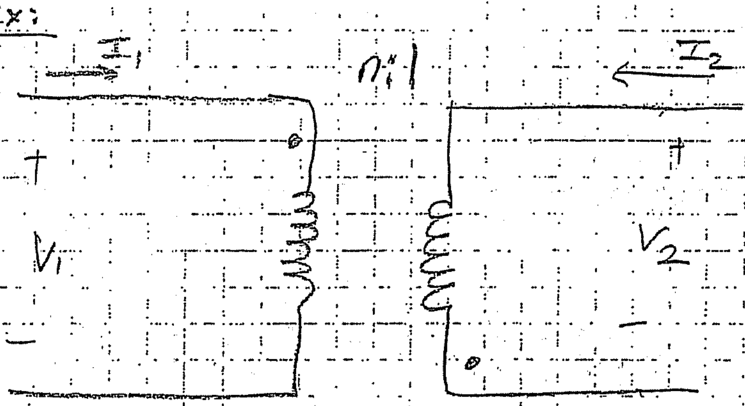
1 L2

Ex:



$$\frac{V_1}{V_2}=n \qquad \frac{I_1}{I_2}=-\frac{1}{n}$$

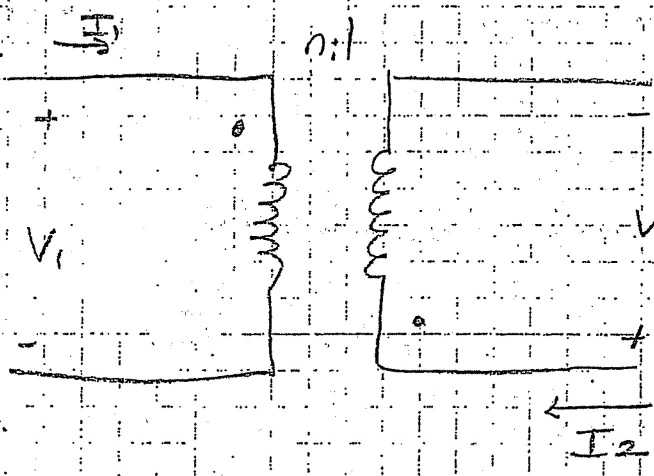
Ex:



$$\frac{V_1}{V_2}=-n$$

$$\frac{I_1}{I_2}=\frac{1}{n}$$

Ex:



$$\frac{V_1}{V_2}=n$$

$$\frac{I_1}{I_2}=-\frac{1}{n}$$

$n>1$ $|V_1|>|V_2|$ step down transformer

primary voltage high

" current low

$n<1$ $|V_1|<|V_2|$ step up transformer

primary voltage low

" current high

* Transformers are used by AC signals.

$P_1(t)$ (primary power) $= V_1(t)\, I_1(t)$

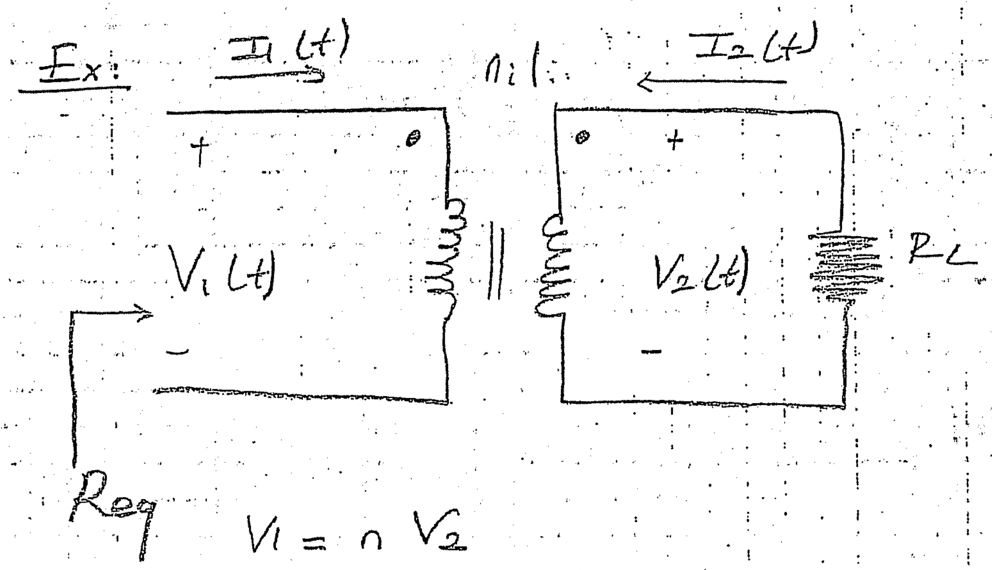
$P_2(t)$ (secondary ") $= V_2(t)\, I_2(t) = \frac{V_1}{n}(-n I_1)$

$= -V_1 I_1$

$P(t) = P_1(t) P_2(t) = 0$ (no power dissipation, " " storage)

Equivalent input resistance:

Ex:



$V_1 = n V_2$

$I_1 = -\frac{1}{n} I_2 \Rightarrow I_2 = -n I_1$

$R_L I_2 = -V_2 \Rightarrow V_2 = -R_L I_2$

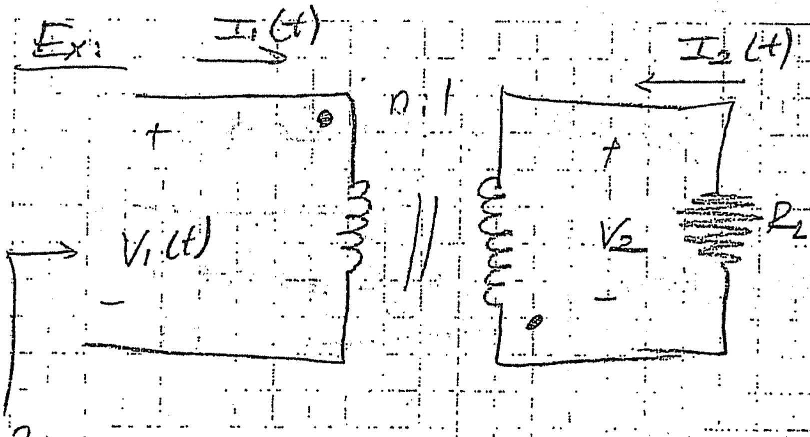
$V_1 = n V_2 = n[-R_L I_2]$

$V_1 = -n R_L I_2$

$V_1 = -n R_L(-n I_1) = n^2 R_L I_1$

$R_{eq} = \frac{V_1}{I_1} = n^2 R_L$

Ex:



Req

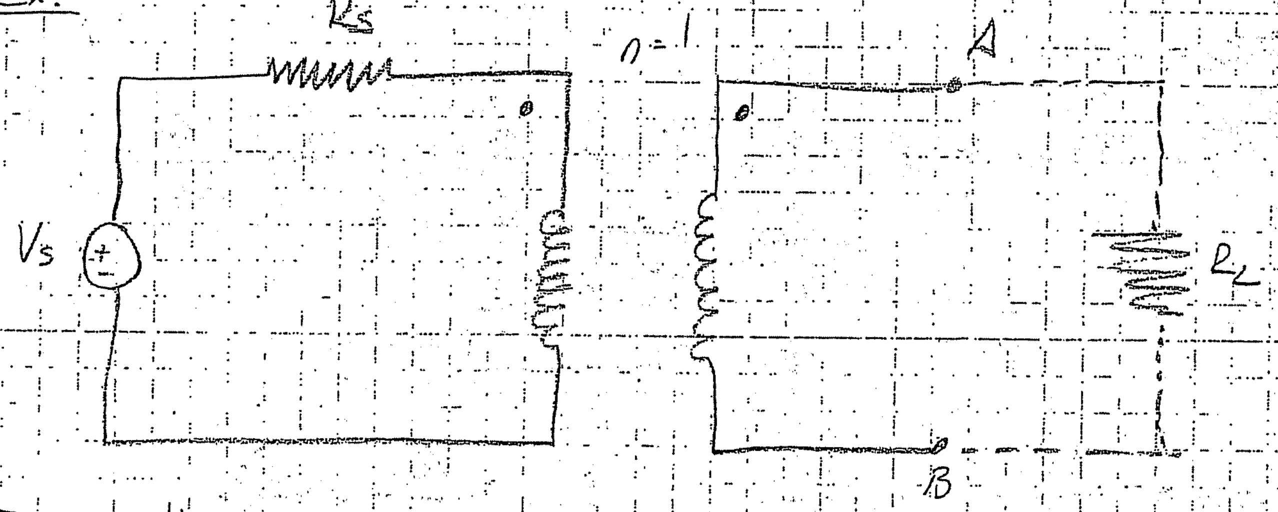
$$V_1 = -n V_2 \qquad \frac{V_2}{R_L} = -I_2$$

$$I_1 = \frac{1}{n} I_2$$

$$V_1 = -n(-I_2 R_L) = -n(-n I_1 R_L) = n^2 I_1 R_L$$

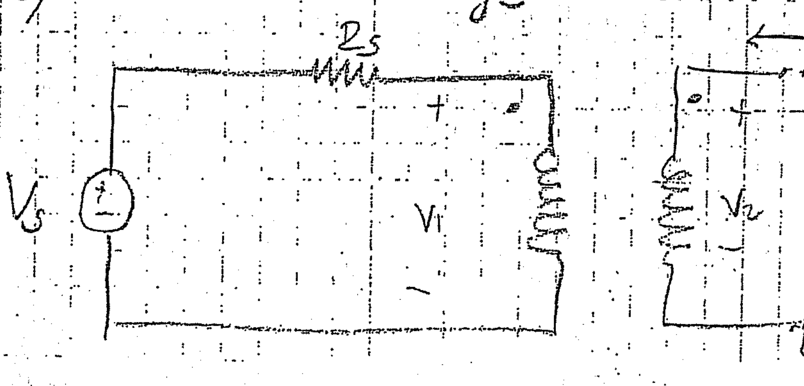
$$R_{eq} = \frac{V_1}{I_1} = n^2 R_L$$

Ex:



Find thevenin equivalent circuit between A-B

open circuit voltage



$V_{AB} = V_{TH}$

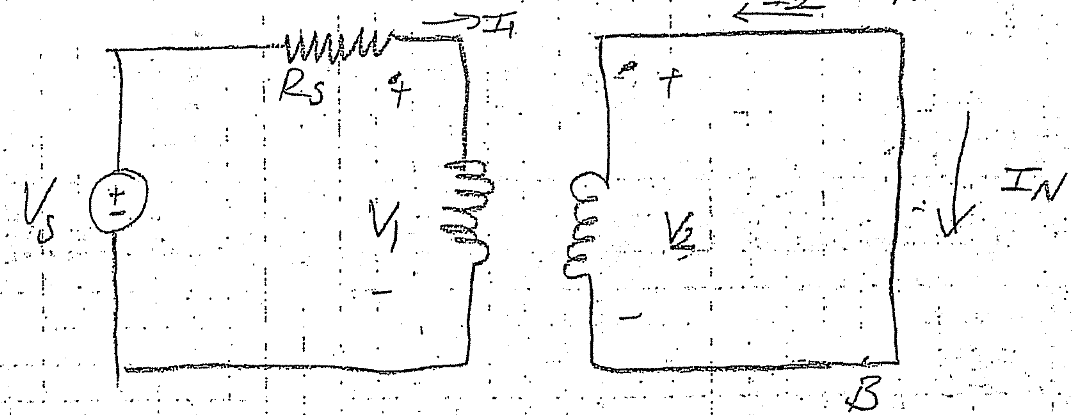
$I_2 = 0 \qquad I_1 = -\frac{1}{n} I_2 = 0$

$V_1 = n V_2 \qquad V_S = V_1 + R_S I_1 = V_1 + R_S * 0$

$V_S = V_1$

$\frac{V_1}{n} = V_2 = \frac{V_S}{n} = V_{AB} = V_{TH} \qquad \boxed{\frac{V_s}{n} = V_{TH}}$

short circuit current I_N



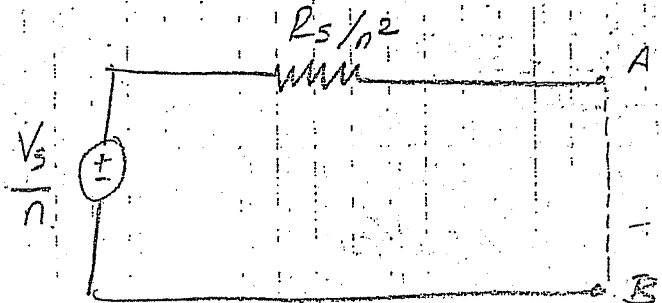
$V_2 = 0 \qquad V_1 = n V_2 = 0$

$I_2 = -n I_1$

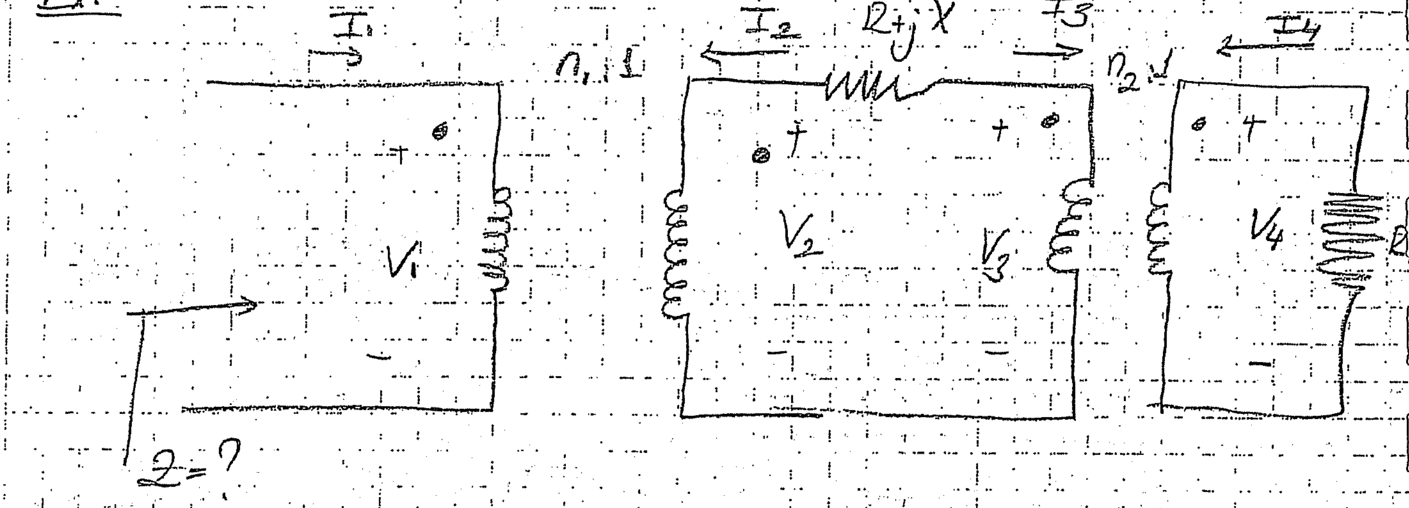
$V_S = I_1 * R_S + V_1 = I_1 R_S + 0 \Rightarrow I_1 = \frac{V_s}{R_s}$

$I_2 = -n \frac{V_S}{R_S} \qquad \boxed{I_N = n \frac{V_S}{R_S}}$

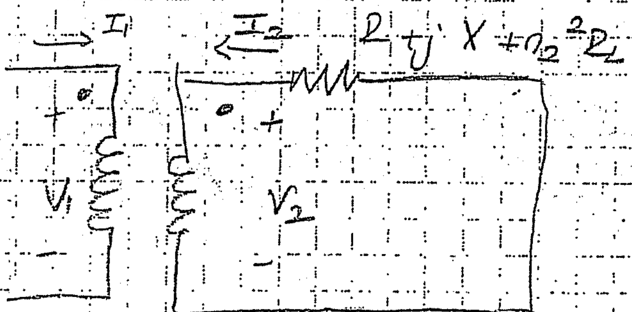
$R_{TH} = \frac{V_{TH}}{I_N} = \frac{V_S/n}{n V_S/R_S} = \frac{\cancel{V_S}}{n} \cdot \frac{R_S}{n \cancel{V_S}} = \frac{R_S}{n^2}$



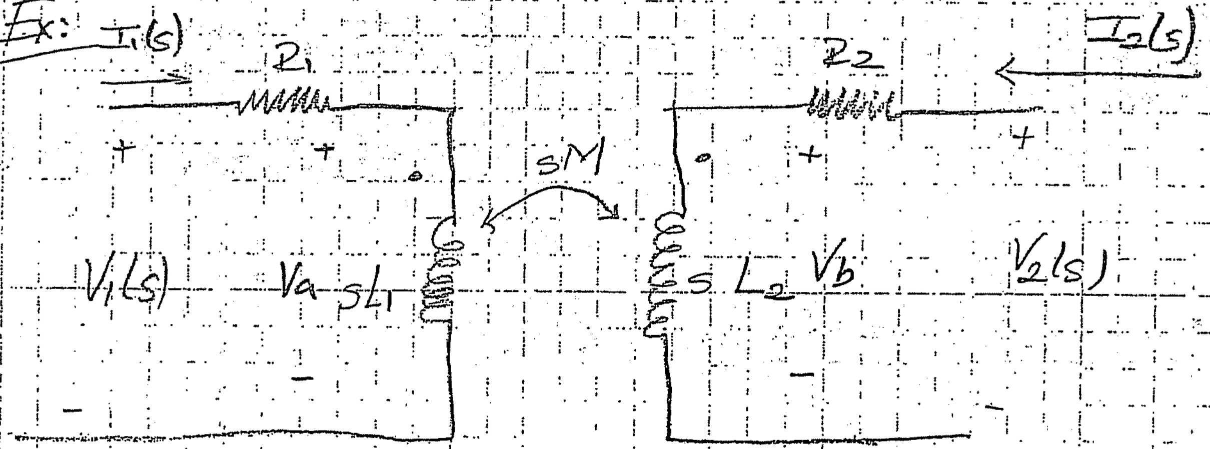
Ex.



$$Z = (n_2^2 R_L + R + jX)\, n_1^2$$



Ex:



Find ABCD parameters in Laplace domain

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} sL_1 & sM \\ sM & sL_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

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$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$V_1 = V_a + I_1 R_1 \Longrightarrow V_1 = sL_1 I_1 + sM I_2 + I_1 R_1$$

$$\boxed{V_1 = [R_1 + sL_1] I_1 + sM I_2}$$

$$V_2 = V_b + R_2 I_2 \Longrightarrow V_2 = sL_2 I_2 + sM I_1 + I_2 R_2$$

$$\boxed{V_2 = [R_2 + sL_2] I_2 + sM I_1}$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{(R_1 + sL_1)\cancel{I_1}}{sM\cancel{I_1}} = \frac{R_1 + sL_1}{sM}$$

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} \qquad V_2 = 0 = (R_2 + sL_2) I_2 + sM I_1$$

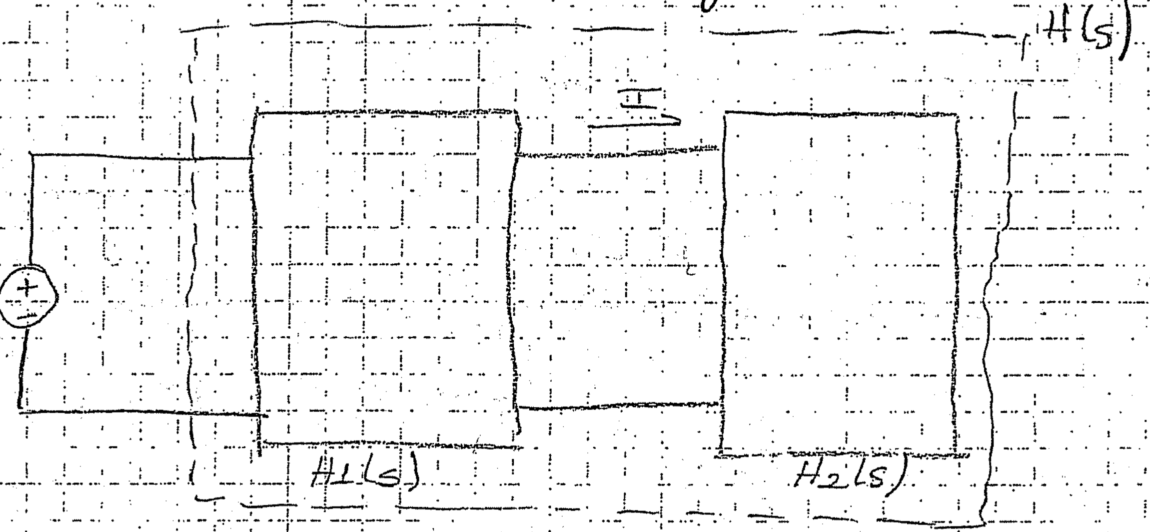
$$I_1 = \frac{-(sL_2 + R_2) I_2}{sM}$$

$$V_1 = \frac{-[R_1 + sL_1](sL_2 + R_2)}{sM} I_2 + sM I_2$$

$$V_1 = \left[\frac{-[R_1 + sL_1][sL_2 + R_2]}{sM} + sM\right] I_2$$

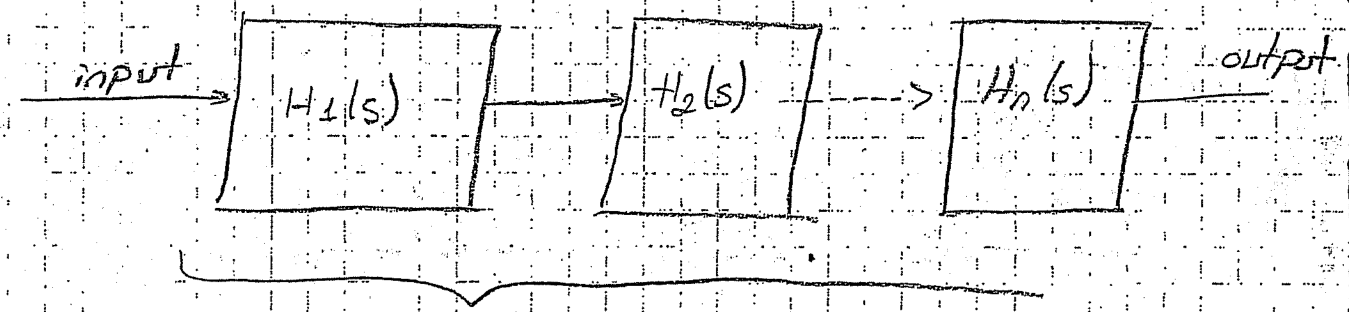
$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = \left[\frac{+[R_1 + sL_1][sL_2 + R_2]}{sM} - sM\right] \qquad C = ? \quad D = ?$$

Cascade Connection and Loading:



if $I = 0$ there is no loading and

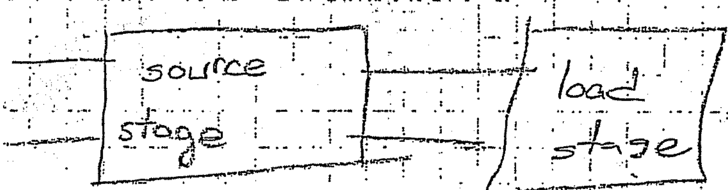
$$\boxed{H(s) = H_1(s) * H_2(s)}$$



$$H_v(s) = H_1(s) \cdot H_2(s) \ldots H_n(s)$$

(Assuming that no loading effect is observed between the stages)

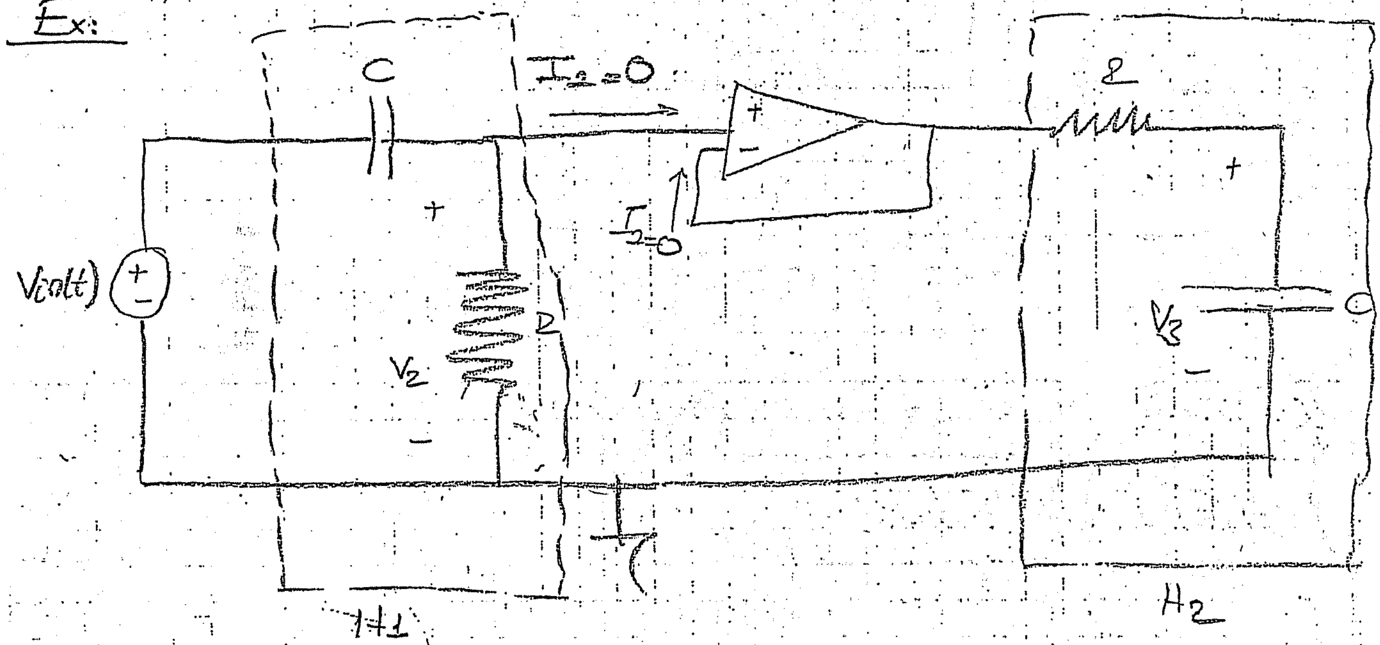
When does loading occurs or not occurs?



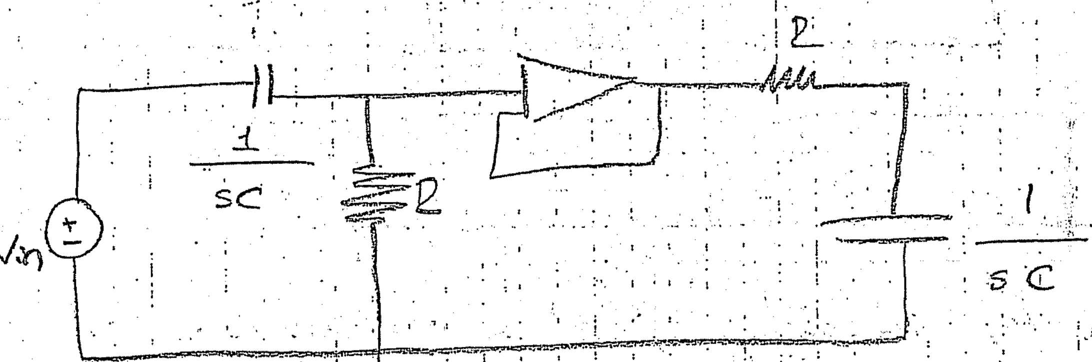
(a) $Z_{TH} = 0$ for source stage
(input impedance $= 0$)

(b) $Z_{in} = \infty$ for load stage

Ex:



$$H_1 = \frac{V_2(s)}{V_{in}(s)}$$



$$H_1 = \frac{R}{\frac{1}{sC} + R} = \frac{RsC}{1 + RsC}$$

(high pass)

$$H_2 = \frac{V_3(s)}{V_2(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1}$$

(low pass)

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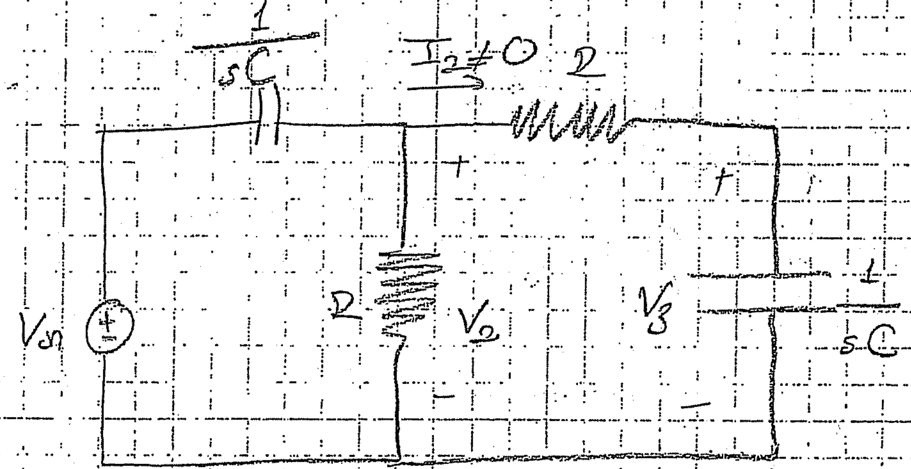
$$\frac{V_3}{V_{in}} = H_1(s).H_2(s) = \frac{RsC}{1+sRC}\cdot\frac{1}{1+sRC} = \frac{sRC}{s^2R^2C^2+2sRC+1}$$

(bandpass)

(no loading)



$$\frac{V_{in}-V_2}{\frac{1}{sC}} = \frac{V_2}{R} + \frac{V_2}{R+\frac{1}{sC}}$$

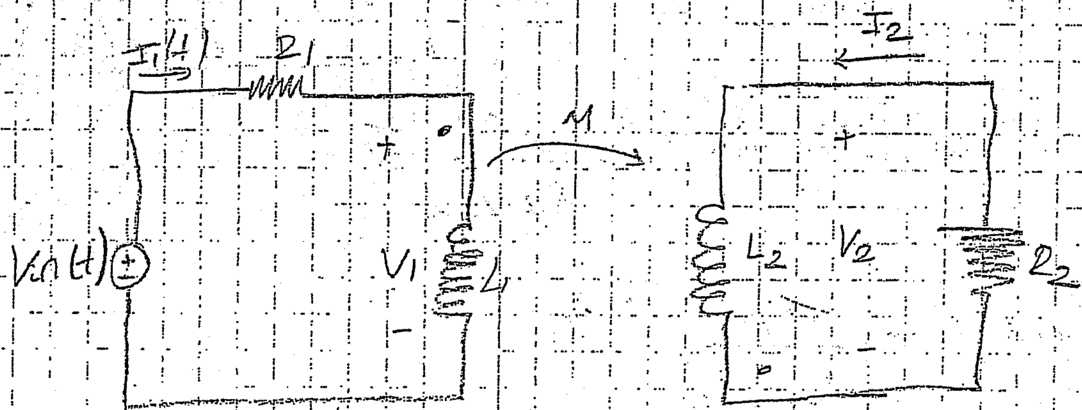
$$\frac{V_2-V_3}{R} = \frac{V_3}{\frac{1}{sC}} \qquad H(s) = \frac{V_3}{V_{in}} = \frac{RCs}{s^2R^2C^2+3sRC+1}$$

(loading effect is observed)

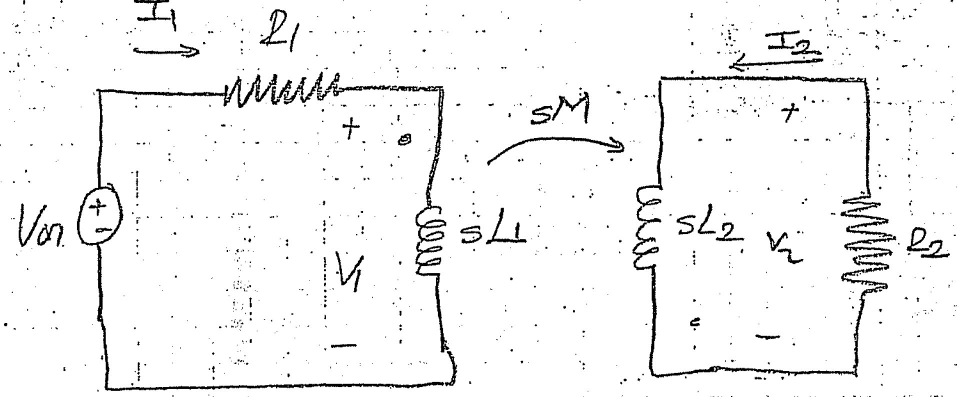
$H(s) \neq H_1(s).H_2(s)$

$H(s) \cong H_1(s).H_2(s)$

Ex:



$$Z_{in}(s) = \frac{V_{in}(s)}{I_1(s)} = \text{Driving point impedance (input " ")}$$



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} sL_1 & -sM \\ -sM & sL_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$V_{in} = R_1 I_1 + V_1 = R_1 I_1 + sL_1 I_1 - sMI_2$$

$$V_2 = -sMI_1 + sL_2 I_2$$

$$V_2 = R_2(-I_2)$$

$$-R_2 I_2 = -sMI_1 + sL_2 I_2$$

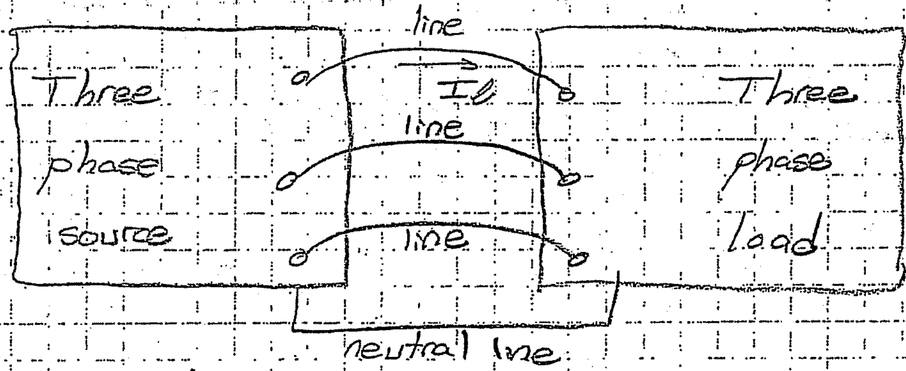
$$sMI_1 = [R_2 + sL_2] I_2 \Rightarrow \boxed{I_2 = \frac{sM}{R_2 + sL_2} I_1}$$

$$V_{in} = [R_1 + sL_1] I_1 - sM \frac{sM}{R_2 + sL_2} I_1$$

$$V_{in} = \left[R_1 + sL_1 - \frac{s^2M^2}{R_2 + sL_2} \right] I_1$$

$$\frac{V_{in}}{I_1} = \frac{\left[R_1+sL_1 - \frac{s^2M^2}{R_2+sL_2}\right]\cancel{I_1}}{\cancel{I_1}} = R_1+sL_1 - \frac{s^2M^2}{R_2+sL_2}$$

Poly-phase circuits:



$I_\ell \rightarrow$ line current

line 1 $\xrightarrow{I_\ell}$ → line current

line 2 $\quad {}^{+}_{-}V_{\ell\ell} = V_\ell \rightarrow$ Line to Line voltage

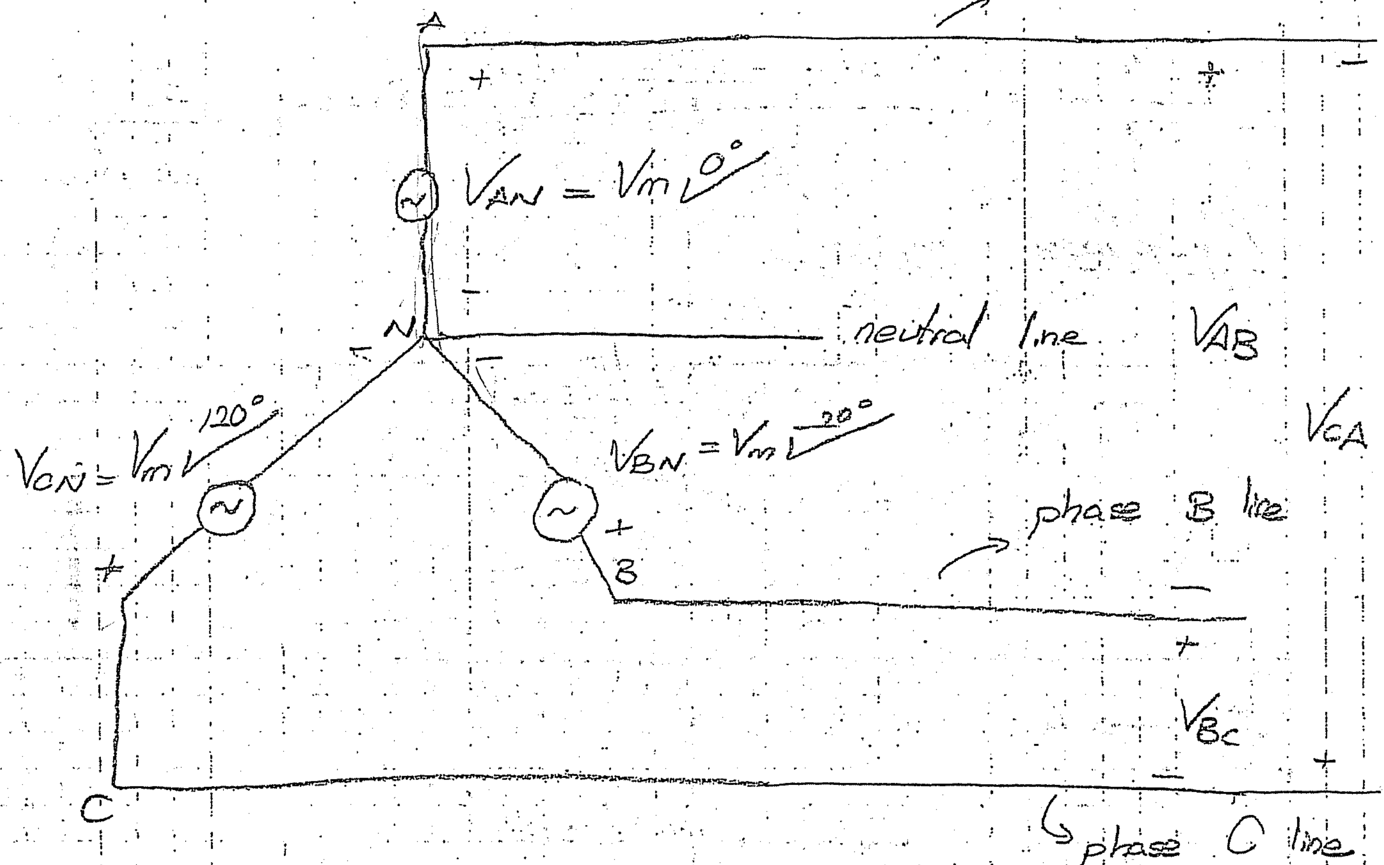
Line 3

${}^{+}_{-}V_p$ = Phase voltage

neutral line

* A balanced three phase voltage is the one which the phase currents or the voltages have equal magnitudes with phase difference of 120°. Source or load sides of the system can be configured to have Δ or Y connection

Y Connection:



$|V_{AN}| = |V_{BN}| = |V_{CN}|$ = phase voltages

$|V_{AB}| = |V_{BC}| = |V_{CA}|$ = line voltages

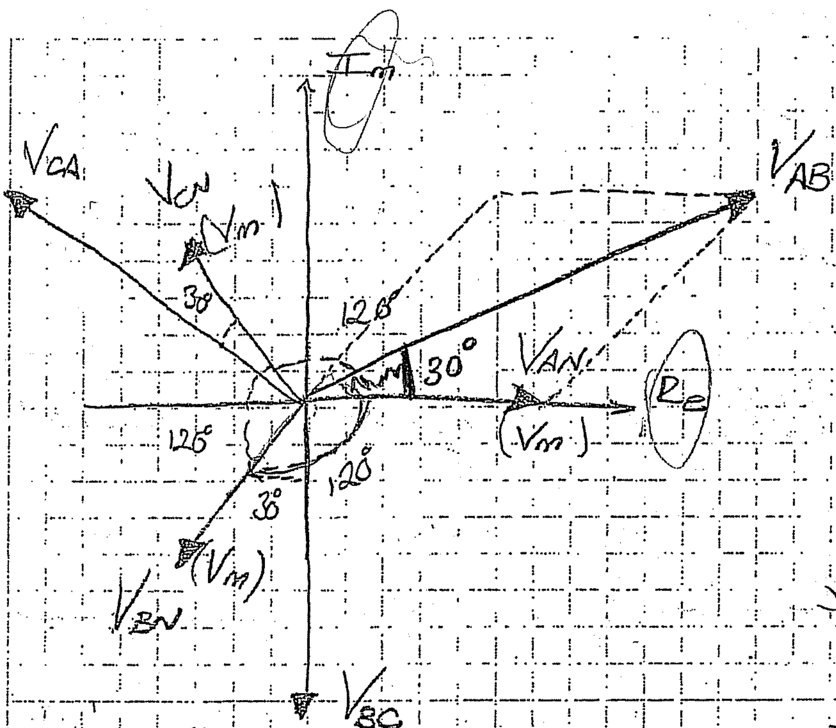
$V_{AN} = V_m \angle 0^\circ$ $\qquad V_{line} = \sqrt{3}\, V_{phase}$

$V_{BN} = V_m \angle -120^\circ$

$V_{CN} = V_m \angle 120^\circ$

$$\boxed{V_{AN} + V_{BN} + V_{CN} = 0 + j0 = 0}$$
(for balance 3 phase system)

3 phase = 3 ϕ



$V_{AB} = V_A - V_B$

$V_m(\cos(0) + j\sin 0) - V_m(\cos(-120) + j\sin(-120))$

phase sequence

V_{AN}; V_{BN}; V_{CN} Line to neutral (phase) voltages

V_{AB}; V_{BC}; V_{CA} line to line voltages

$$V_{AB} = V_A - V_B = V_m\angle 0^\circ - V_m \angle 120^\circ$$

$$= V_m(\cos 0 + j\sin 0) - V_m(\cos(120) + j\sin(-120))$$

$$= V_m - V_m\left[-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right]$$

$$= V_m + \frac{V_m}{2} + \frac{V_m\sqrt{3}}{2}j$$

$$= \frac{3}{2}V_m + \frac{\sqrt{3}}{2}V_m j$$

$$\arctan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$$

$$\text{mag} = \sqrt{\left(\frac{3}{2}\right)^2 V_m^2 + \left(\frac{\sqrt{3}}{2}\right)^2 V_m^2} = \sqrt{3}\,V_m$$

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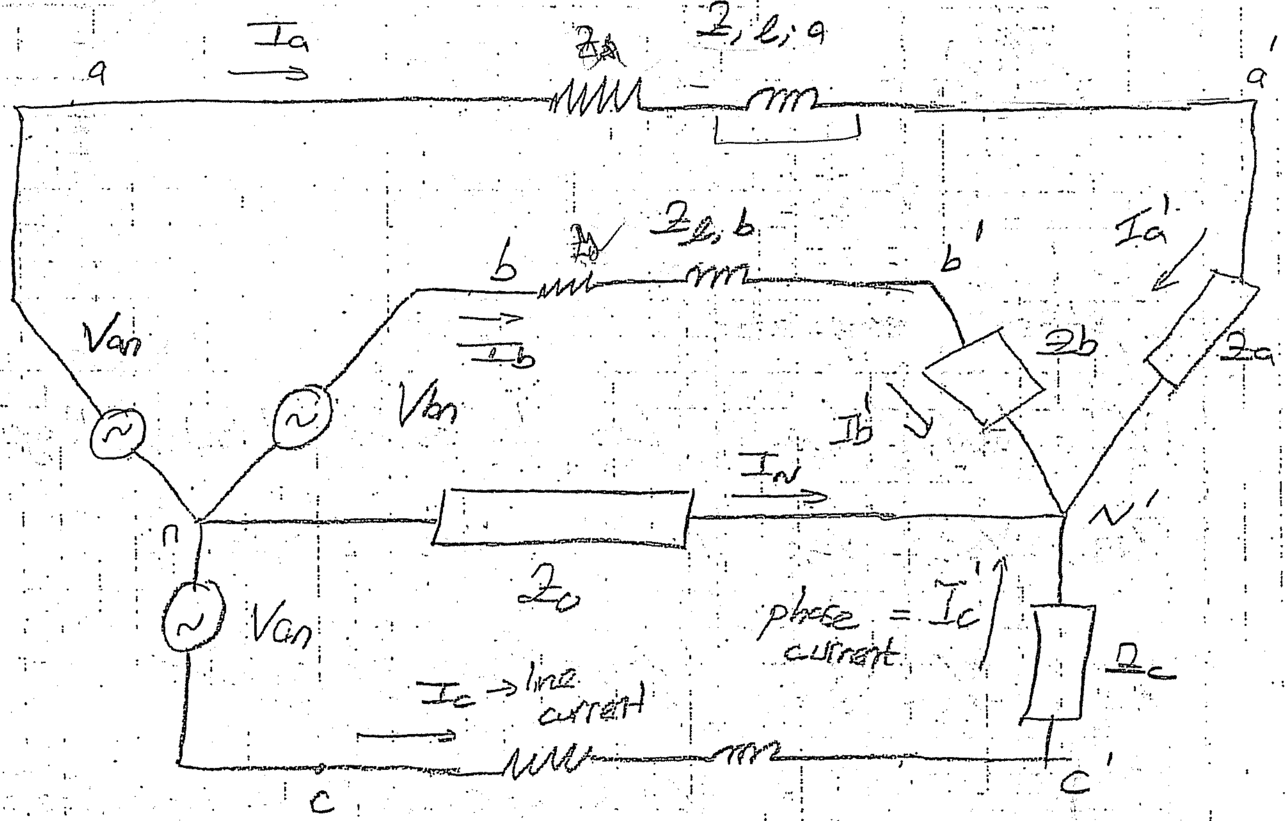
$$\boxed{V_{AB} = \sqrt{3}\, V_m \angle 30^\circ}$$

$$V_{CA} = \sqrt{3}\, V_m \angle 150^\circ$$

$$V_{BC} = \sqrt{3}\, V_m \angle -90^\circ$$

$$|V_{AB}| = |V_{CA}| = |V_{BC}| = |V_{line}| = |V_{line\text{-}line}|$$



$Z_{l,a}$; $Z_{l,b}$; $Z_{l,c}$ $\longrightarrow$ transmition line impedances

To have a balanced 3ϕ system

① $|V_{an}| = |V_{bn}| = |V_{cn}|$ and they should form a balanced 3ϕ source configuration

② Transmition lines should be balanced

$$Z_{la} = Z_{lb} = Z_{lc}$$

(3) Load should be balanced

$$Z_a = Z_b = Z_c$$

* If this system is balanced; all phase currents are equal in magnitude with 120° phase difference

$$|I_a| = |I_b| = |I_c| \longrightarrow \text{line current}$$

$$|I_a'| = |I_b'| = |I_c'| \longrightarrow \text{phase current}$$

$$|I_a| = |I_a'|$$

$$I_a + I_b + I_c = I_N$$

$$\frac{V_{an}}{Z_{la} + Z_a} + \frac{V_{bn}}{Z_{lb} + Z_b} + \frac{V_{cn}}{Z_{lc} + Z_c} = I_N$$

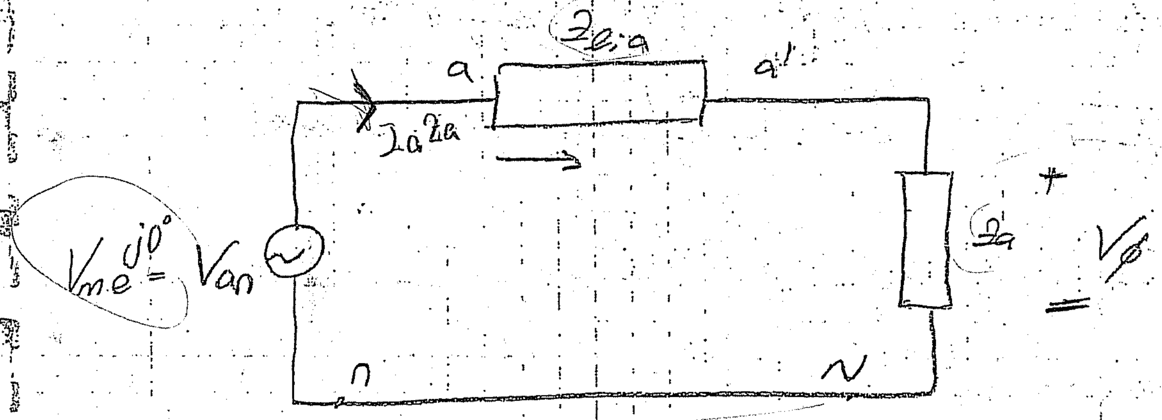
$$Z = Z_{la} + Z_a = Z_{lb} + Z_b = Z_{lc} + Z_c$$

$$\Longrightarrow \frac{V_{an}}{Z} + \frac{V_{bn}}{Z} + \frac{V_{cn}}{Z} = I_N$$

$$\frac{1}{Z}\Big[\underbrace{V_{an} + V_{bn} + V_{cn}}_{0}\Big] = I_N = 0$$

Single Phase Approach;

For a balanced Y-Y connected system; we need to consider only one phase circuit to find all the currents and voltages.



$$I_a = \frac{V_m e^{j0°}}{Z_{li;a} + Z_a} \quad \text{Line current}$$

$$\boxed{V_\phi = I_a Z_a = V_m e^{j0°} \frac{Z_a}{Z_a + Z_{li;a}}}$$

phase voltage

$$|V_{Line-Line}| = \sqrt{3}\,|V_\phi|$$

Let $V_\phi = V_{a'n}$ as the reference

$$\left.\begin{aligned} V_{a'n} &= V_\phi \angle 0° \\ V_{a'b'} &= \sqrt{3}\, V_\phi \angle 30° \\ V_{b'c'} &= \sqrt{3}\, V_\phi \angle -90° \\ V_{c'a'} &= \sqrt{3}\, V_\phi \angle 150° \end{aligned}\right\} \text{line voltages at the load side}$$

Power (assuming RMS values)

$$P_p = Re\{V_a I_a^*\} = Re\{Z_a I_a I_a^*\}$$

Average power for a single phase at the load $= |I_a|^2 Re\{Z_a\}$

$$Q_p = Im\{V_p I_a^*\} = Im\{Z_a I_a I_a^*\}$$

Reactive power for a single phase at the load $= |I_a|^2 Im\{Z_a\}$

$|S_p| = |V_p| |I_a|$ $\quad S_p = P_p + jQ_p$

↳ appearent power

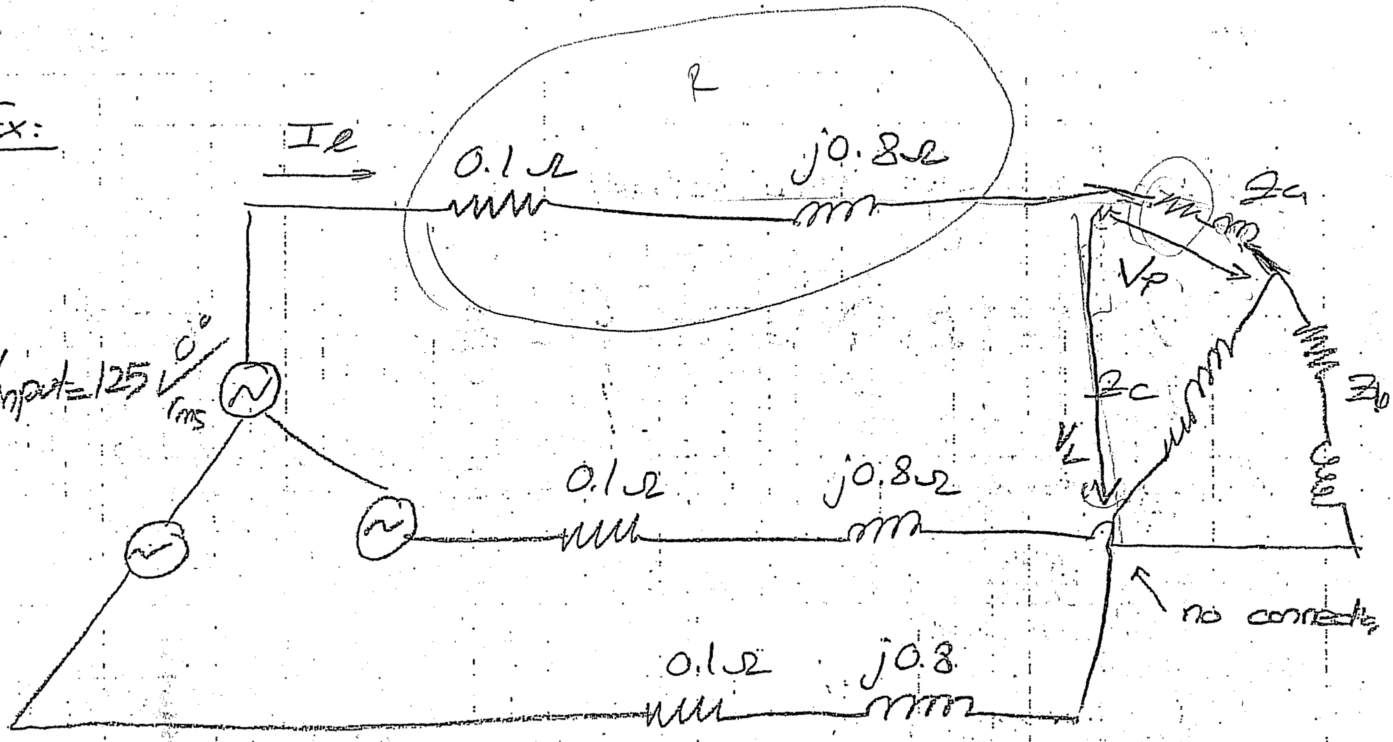
overall average power at load is $P_{Total} = 3 P_p$ (phase power)

$$P_{Total} = P_{3\phi} = 3 |I_a|^2 Re\{Z_a\}$$

$$Q_{Total} = Q_{3\phi} = 3 |I_a|^2 Im\{Z_a\}$$

$$|S_{3\phi}| = |S_{Total}| = 3 |I_a| |V_p|$$

Ex:

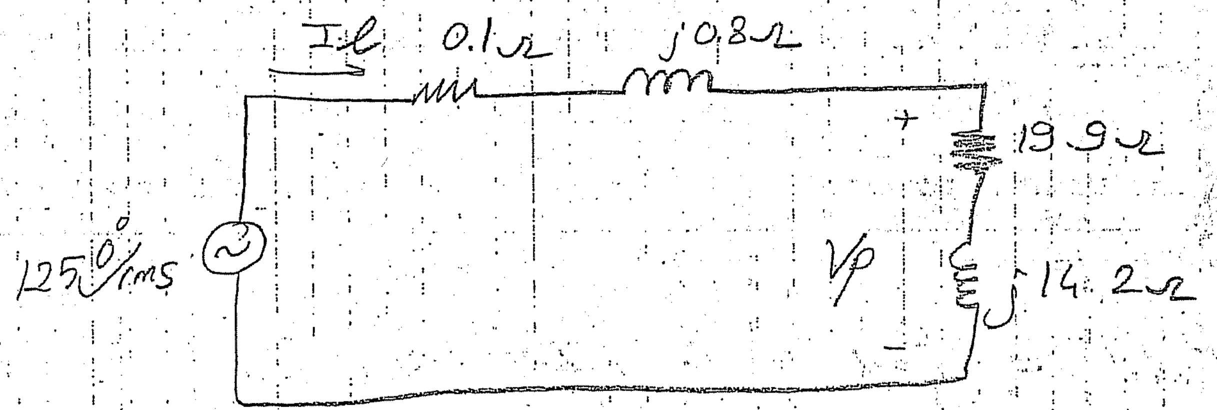


$Z_a = Z_b = Z_c = [19.9 + j14.2]\ \Omega$

(a) Find I_l; V_L; V_p at load side

(b) Find the total 3Ø average and reactive absorbed at the load

Use single phase approach.



$$I_l = \frac{125\angle 0°}{0.1 + j0.8 + 19.9 + j14.2} = \frac{125\,(\overbrace{\cos 0 + j\sin 0°}^{1})}{20 + 15j}$$

$$= \frac{125}{20 + 15j} = \frac{25}{4+3j}\frac{(4-3j)}{(4-3j)} = \frac{100 - 75j}{16 + 9} = \frac{100-75j}{25} = 4 - 3j$$

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$\sqrt{16+9}$

$I_\ell = 4 - 3j = 5\angle{-37^\circ}$ Amper (rms)

$V_p = I_\ell\,(19.9 + j\,14.2)$

$V_p = 5\angle{-37^\circ}\,[19.9 + j\,14.2] = 122.2\angle{-1.5^\circ}$ Volt (rms)

$4-3j$

$V_L = \sqrt{3}\,|V_p| = 211.7$ Volt (rms)

$(4-3i)(19.9 + j14.2)$

(b) $P_{3\phi} = 3\,|I_\ell|^2\,Re\{Z_a\}$ $\quad 79.6 + 56.8j - 7$

$122.2 - 2.9j$

load $= 3\,|5|^2\,(19.9)$ Watt

$Q_{3\phi} = 3\,|I_\ell|^2\,Im\{Z_a\}$

load $= 3\,|5|^2\,(14.2)$ VAR

* For a balanced three phase system, total instantaneous power is constant and it is = 3 * average power per phase

$V_a(t)\,;\ I_a(t)$
$V_b(t)\,,\ I_b(t)$ } instantaneous values
$V_c(t)\,;\ I_c(t)$

$P(t) = P_1(t) + P_2(t) + P_3(t)$

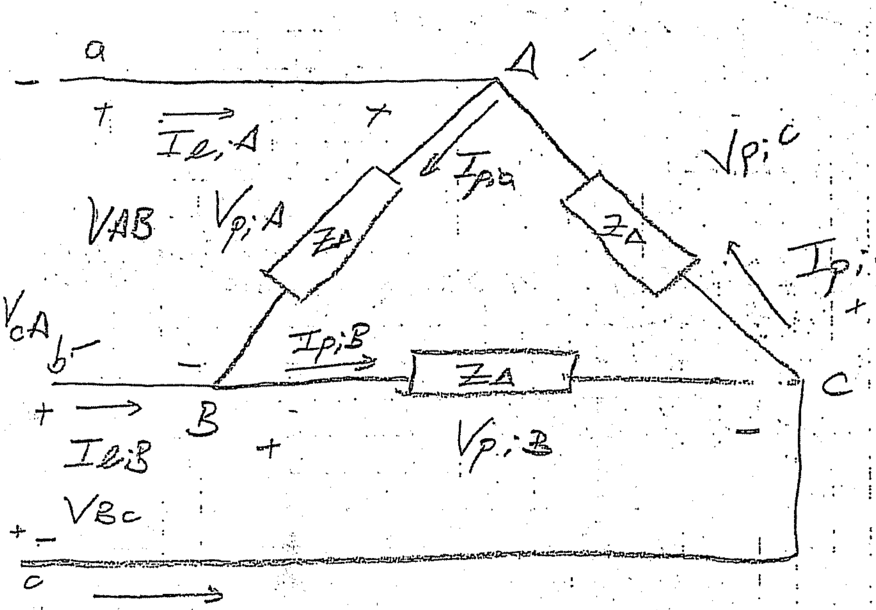
instantaneous power $= V_1(t)\,I_1(t) + V_2(t)\,I_2(t) + V_3(t)\,I_3(t)$

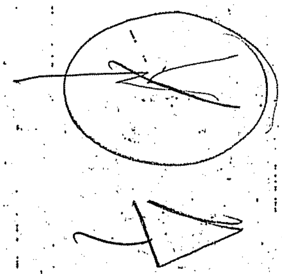
$P_T = 3\,V_\phi I_\phi \cos\theta = 3 * P_p$

Total average power $\qquad P_T = P(t)$

HOMEWORK: Show that

Δ - Connection:





When the load side is balanced

$$Z_\Delta = 3 Z_Y$$

$$\left.\begin{aligned} V_{ab} &= V_{p;A} \\ V_{bc} &= V_{p;B} \\ V_{ca} &= V_{p;C} \end{aligned}\right\}$$ line and phase voltages are equal

The phase currents

$$I_{p;A} = I_\phi \angle 0^\circ$$

$$I_{p;B} = I_\phi \angle -120^\circ$$

$$I_{p;C} = I_\phi \angle +120^\circ$$

The line currents

$$I_{l;A} = I_{p;A} - I_{p;C}$$

$$= I_\phi \angle 0^\circ - I_\phi \angle 120^\circ$$

$$= \sqrt{3} I_\phi \angle -30^\circ$$

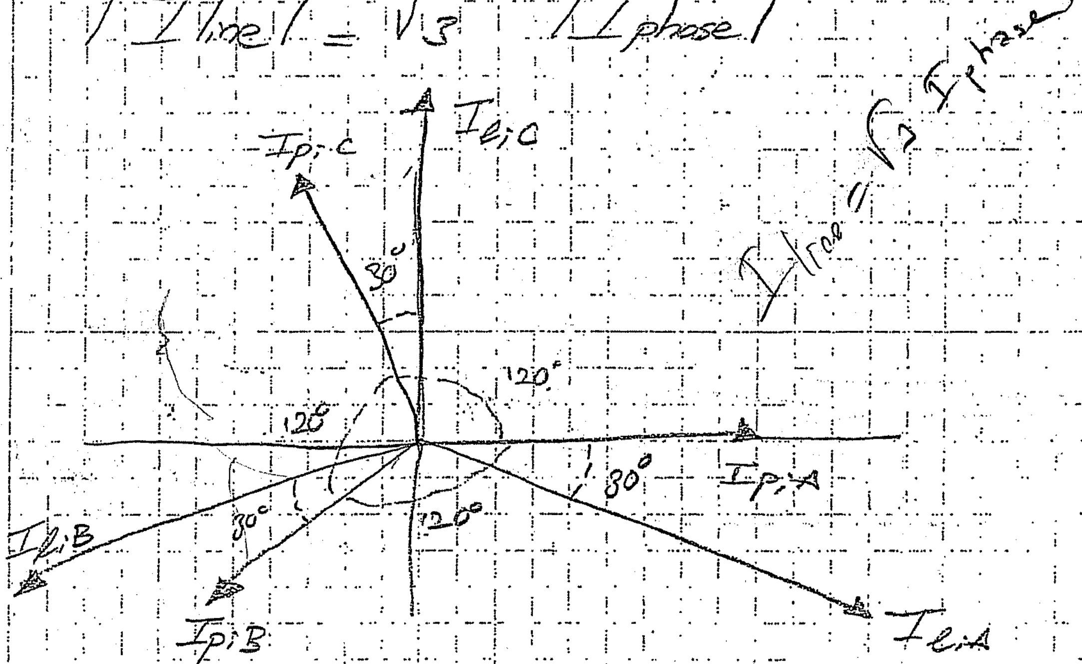
$$I_{l;B} = \sqrt{3} I_\phi \angle -150$$

$$I_{l;C} = \sqrt{3} I_\phi \angle 90^\circ$$

$|I_{l,A}| = |I_{l,B}| = |I_{l,C}| = |I_{line}|$

$|I_{p,A}| = |I_{p,B}| = |I_{p,C}| = |I_{phase}|$

$|I_{line}| = \sqrt{3}\ |I_{phase}|$



Complex Power:

$S_1 = V_{ab}\ I_{p,A}^{*}$

$= |V_{line}|\ |I_{phase}| \angle\theta$ rms value

$S_2 = V_{bc}\ I_{p,B}^{*} \angle\theta$

$S_3 = V_{ca}\ I_{p,C}^{*} \angle\theta$

$S_T = S_1 + S_2 + S_3 = 3\ |V_{line}|\ |I_{phase}| \angle\theta$

$S_{Total} = 3\ |V_{line}|\ |I_{phase}| \angle\theta$

$S_{total} = \sqrt{3}\ |V_{line}|\ |I_{line}| \angle\theta$

} Δ Connection

$|V_{line}| = |V_{phase}|$

$|I_{phase}| = \frac{|I_{line}|}{\sqrt{3}}$

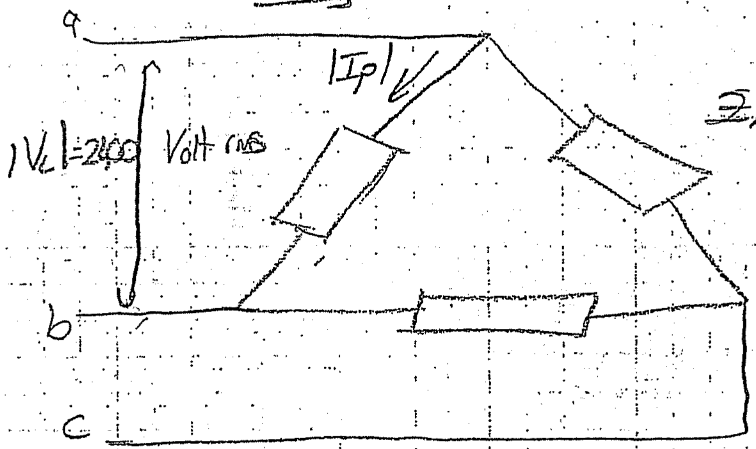
$$S_{Total} = 3\ |V_{phase}|\ |I_{line}| \angle\theta$$
$$S_{Total} = \sqrt{3}\ |V_{line}|\ |I_{line}| \angle\theta$$

Y Connection

$|I_{line}| = |I_{phase}|$

$$|V_{phase}| = \frac{|V_{line}|}{\sqrt{3}}$$

Ex: $|I_L|$



$Z_\Delta = 40 + 30j = 50\ e^{j36.87^\circ}$

Find the line current

pf = ?

Total P, Q, S for load

$|V_L| = 2400$

$$|I_p| = \frac{2400}{50} = \frac{|V_L|}{|Z_\Delta|} = 48\ \text{Amp}$$

$pf = P/S$

$$pf = \cos\theta = \frac{Re\{Z_\Delta\}}{|Z_\Delta|} = \frac{40}{50} = 0.8$$

$$|I_L| = \sqrt{3}\ |I_p| = \sqrt{3} * 48$$

$$P_\phi = |I_p|^2 Re\{Z_\Delta\} = 48^2 * 40$$

P_ϕ → single phase

$$P_{3\phi} = 3\ |I_p|^2 Re\{Z_\Delta\} = 3 * 48^2 * 40 = 276480\ \text{Watts}$$

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$$Q_{3\phi} = 3 * |I_p|^2 * Im\{Z_\Delta\} = 207360 \text{ VAR}$$

$$S_{3\phi} = P_{3\phi} + j\,Q_{3\phi} = 276480 + j\,207360 \text{ VA}$$

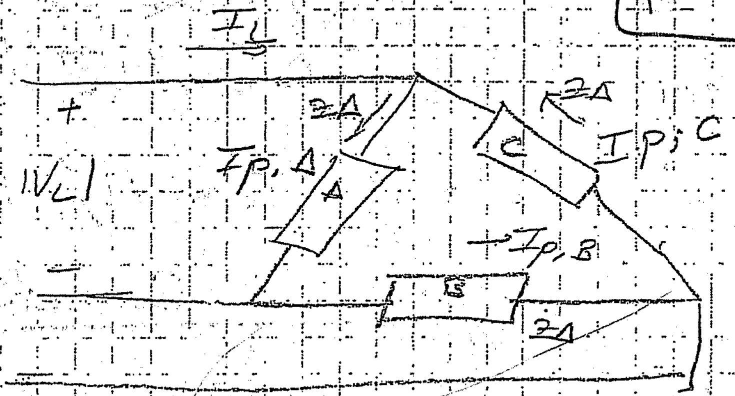
$$|S_{3\phi}| = 345600 \text{ VA}$$

Complex Power

$$S_{3\phi} = P_{3\phi} + j\,Q_{3\phi}$$

$$S_{\phi} = P_{\phi} + j\,Q_{\phi}$$

$$\boxed{pf = \cos\theta = \frac{P}{|S|}}$$



$$S_1 = V_L * I_{p,A}^* = Z_\Delta\, I_{p,A} * I_{p,A}^*$$

$$= Z_\Delta\, |I_{p,A}|^2$$

$$= \left[Re\{Z_\Delta\} + j\, Im\{Z_\Delta\}\right] |I_{p,A}|^2$$

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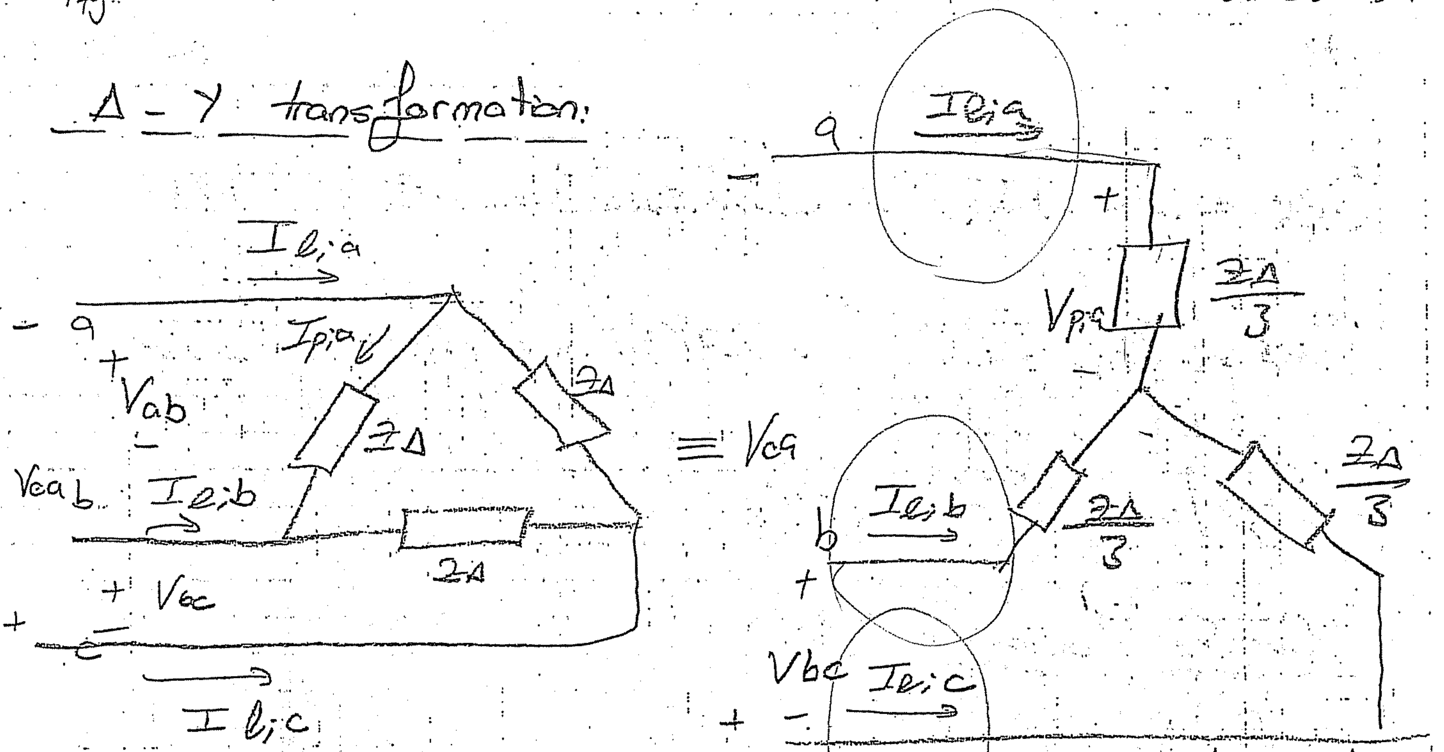
Δ - Y transformation:



$|I_{l;c}| = |I_{la}| = \sqrt{3}\,|I_{p;a}| = |I_{l;b}|$

$V_l = |V_{ab}| = |V_{bc}| = |V_{ca}|$

$|S| = \sqrt{3}\,|V_l|\,|I_{l;a}|$

$|S| = 3\,|V_l|\,|I_{p;a}|$

$|V_{p;a}| = \frac{|V_{l;a}|}{\sqrt{3}} = \frac{|V_l|}{\sqrt{3}}$

$\boxed{|S| = \sqrt{3}\,|I_{l;a}|\,|V_L|}$

$= \sqrt{3}\,|I_{l;a}|\,\sqrt{3}\,|V_{p;a}|$

$|S| = 3\,|I_{l;a}|\,|V_{p;a}|$

Y-connection (load)

$P_\phi = |V_p|\,|I_p|\,\cos\theta$

$P_\phi = \frac{|V_l|}{\sqrt{3}}\,|I_l|\,\cos\theta$

$P_T = \sqrt{3}\,|V_L|\,|I_L|\,\cos\theta$

$V_l = \sqrt{3}\,|V_p|$

$Q_T = \sqrt{3}\,|V_L|\,|I_L|\,\sin\theta$

$S_T = P_T + jQ_T$

Δ - Connection (load)

$P_\phi = |V_p|\,|I_p|\,\cos\theta$

$P_\phi = |V_l|\,\frac{|I_l|}{\sqrt{3}}\,\cos\theta$

$I_l = |I_p|\,\sqrt{3}$

$P_T = \sqrt{3}\,|V_l|\,|I_l|\,\cos\theta$

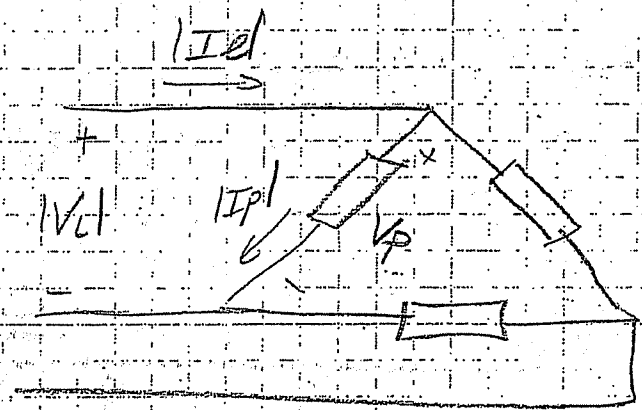
$Q_T = \sqrt{3}\,|V_l|\,|I_l|\,\sin\theta$

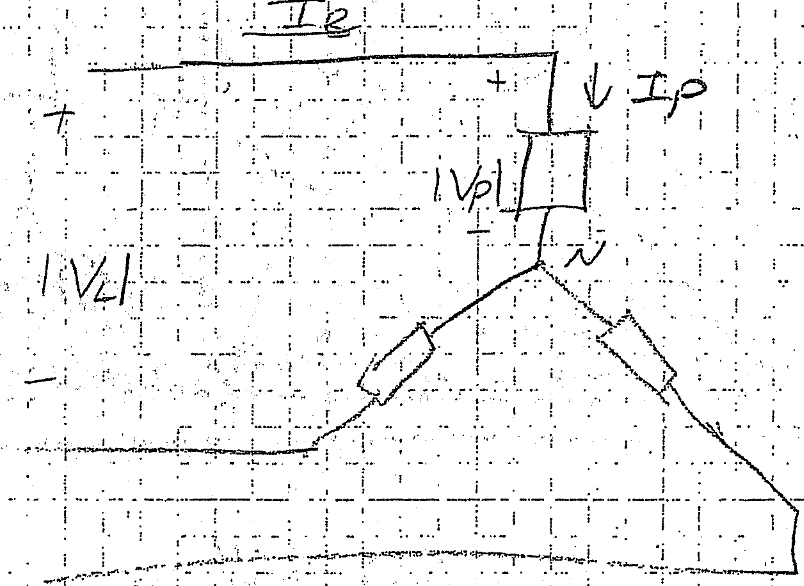
$S_T = P_T + jQ_T$

just remember Δ connected load

$$|I_\ell| = \sqrt{3}\,|I_p|$$

$$|V_\ell| = |V_p|$$





$$|V_\ell| = \sqrt{3}\,|V_p|$$

$$|I_\ell| = |I_p|$$

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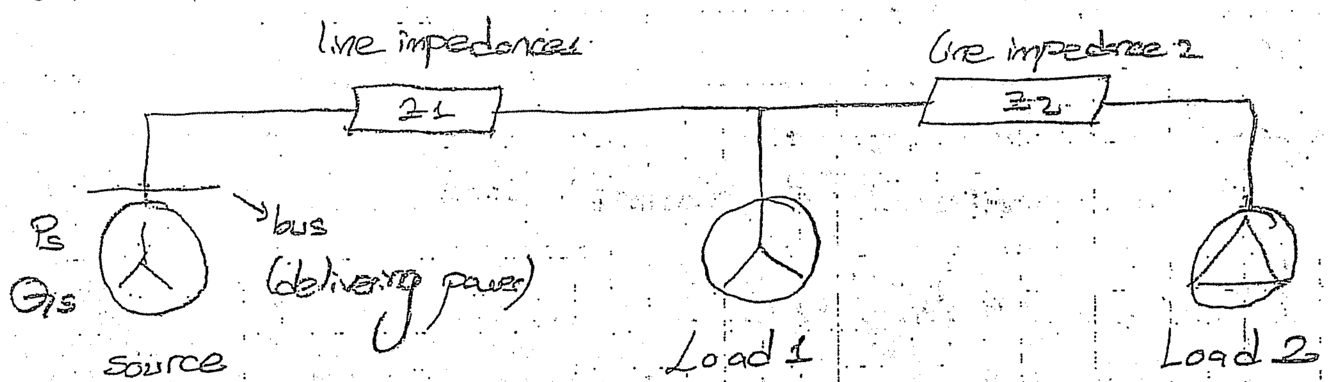
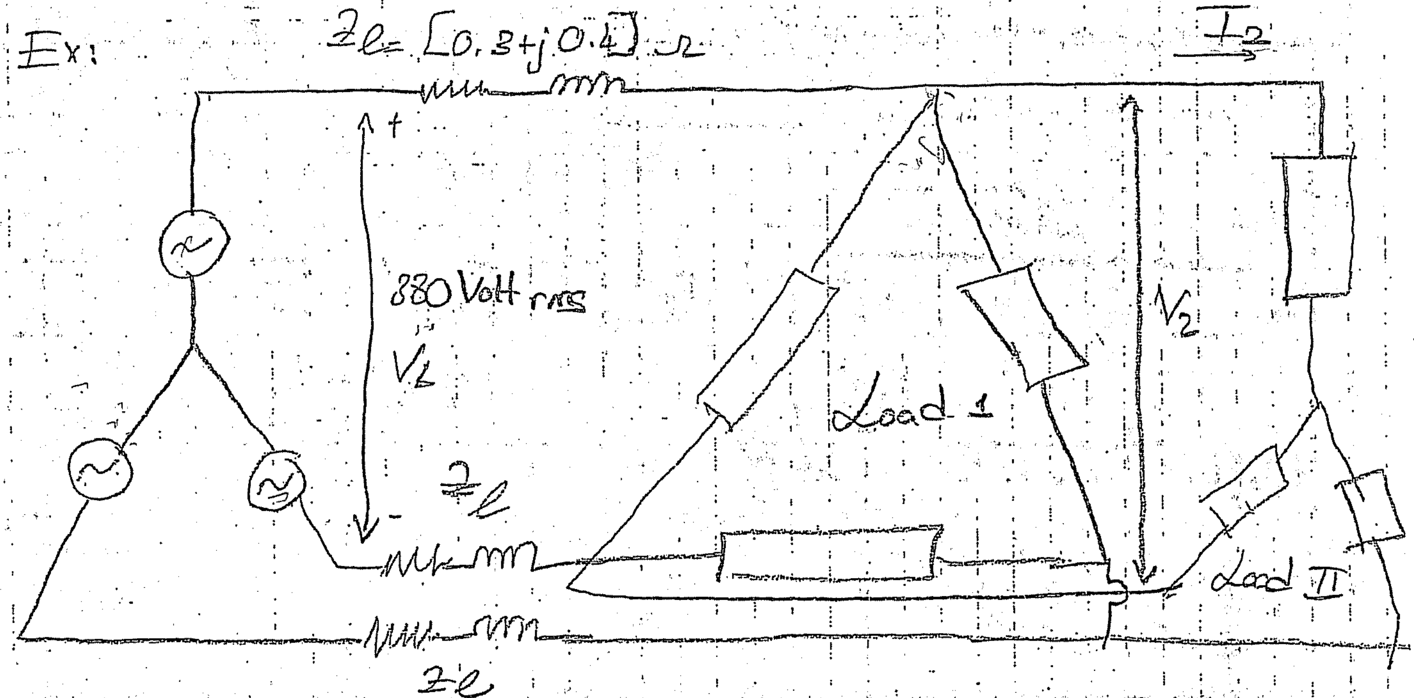


single line representation

- the neutral line is omitted (not shown)
- the currents and voltages associated with different phases are shown only by their magnitude.

Ex:



* Source side line voltage = $V_L = 380$ Volt rms

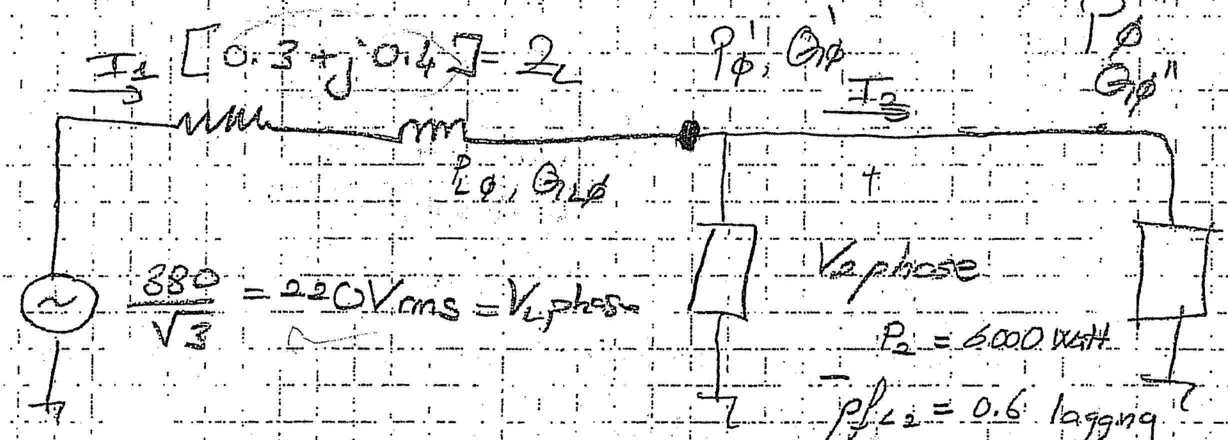
* Source delivers a total of 9 kW and 9 kVAR

Load I: absorbs 6 kW at a 0.6 pf of lagging

Find $|V_2|$; $|I_2|$; $[P_2, Q_2$ and $pf_2] \rightarrow$ for load II

1st approach:

1^{st} approach: Single-phase approach:



$\frac{9}{3}kW + \frac{9}{3}j\,kVAR$ complex power delivered per phase

$P_{S\phi} = \frac{9}{3} = 3\,kW \qquad P_{S\phi} = 3000$

$Q_{S\phi} = \frac{9}{3} = 3\,kVAR \qquad Q_{S\phi} = 3000$

$S_{S\phi} = P_{S\phi} + j\,Q_{S\phi} = 3000 + 3000j$

$|S_{S\phi}| = |V_{L\,phase}|\,|I_1|$

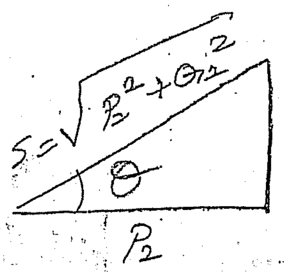
$\sqrt{3000^2 + 3000^2} = 220\,|I_L| \Rightarrow |I_L| = 19.3\ A$

$P_{L\phi} = |I_1|^2\,Re\{Z_L\} = |19.3|^2 * 0.3 = 112$ Watts

$Q_{L\phi} = |I_1|^2\,Im\{Z_L\} = |19.3|^2 * 0.4 = 149$ VAR

$P_\phi' = 3000 - P_{L\phi} = 3000 - 112 = 2888$ Watt

$Q_\phi' = 3000 - Q_{L\phi} = 3000 - 149 = 2851$ VAR



$$pf = \cos\theta = 0.6 = \frac{P_2}{\sqrt{P_2^2 + Q_2^2}}$$

$$\cos\theta = \frac{6000}{\sqrt{6000^2 + Q_2^2}} = 0.6$$

$\Rightarrow Q_2 = 8000$

$P_{2\phi} = \frac{6000}{3} = 2000$ Watt

$Q_{2\phi} = \frac{8000}{3} = 2666$ VAR

$S_\phi' = \sqrt{(2888)^2 + (2851)^2} = 4058$ VA

$|I_1|\,|V_2 phase| = S_\phi'$

$19.3\ |V_2 phase| = 4058 \Rightarrow |V_2 phase| = 210.3$ Voltsrms

$V_2 = \sqrt{3} * 210.3 = \underline{364}$ Volts

$P_\phi'' = P_\phi' - P_{2\phi} = 2888 - 2000 = 888$ Watt

$Q_\phi'' = Q_\phi' - Q_{2\phi} = 2851 - 2666.6 \cong 187$ VAR

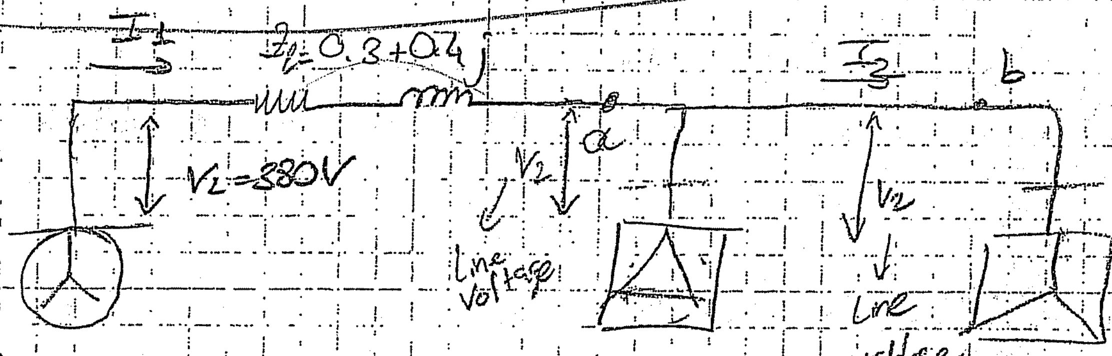
$|S_\phi''| = \sqrt{(P_\phi'')^2 + (Q_\phi'')^2} = \sqrt{888^2 + 187^2} = 907$

$|S_\phi''| = |I_2|\,|V_2 phase|$

$907 = |I_2|\ 210.3 \Rightarrow |I_2| = 4.31$ A

$$pf_{Load2} = \frac{P_\phi''}{|S_\phi''|} = \frac{888}{907} = 0.98 \text{ lagging}$$

2^{nd} approach (Single line)



$P_S = 9kW$ $\qquad$ $P_2 = 6kW$

$Q_S = 9kVAR$ $\qquad$ 0.6 pf lagging

S_S $\qquad$ $Q_2 = 6\tan^{-1}[\cos(0.6)] = 7992 VAR$

Consider the power

$$S_S = \sqrt{P_S^2 + Q_S^2} = \sqrt{9^2 + 9^2} = 12.72 \text{ kVA}$$

$$|I_L| = \frac{|S_S|}{V_L\sqrt{3}} = \frac{12.72 \cdot 10^3}{\sqrt{3} * 380} = 19.3 \text{ Amp}$$

$$P_{Transfer} = |I_L|^2 \, 0.3 * 3 = 336 \text{ Watt}$$

$$Q_{Transfer} = |I_L|^2 \, 0.4 * 3 = 447 \text{ VAR}$$

$$P_a = P_S - P_{Transfer} = 9000 - 336 = 8664 \text{ Watt}$$

$$Q_a = Q_S - Q_{Transfer} = 9000 - 447 = 8553 \text{ VAR}$$

$$|S_a| = \sqrt{P_a^2 + Q_a^2} = 12174.5 \text{ VA}$$

$$|V_2| = \frac{|S_a|}{|I_1|\sqrt{3}} = \frac{12174.5}{19.3 * \sqrt{3}} = 364 \text{ Volts}$$

$$P_b = P_a - P_2 = 8664 - 6000 = 2664 \text{ Watts}$$

$$Q_b = Q_a - Q_2 = 8553 - 7992 = 561 \text{ VAR}$$

$$|S_b| = \sqrt{P_b^2 + Q_b^2} = \sqrt{2664^2 + 561^2} = 2722.4 \text{ VA}$$

$$|I_2| = \frac{|S_b|}{\sqrt{3}|V_2|} = 4.31 \text{ A}$$

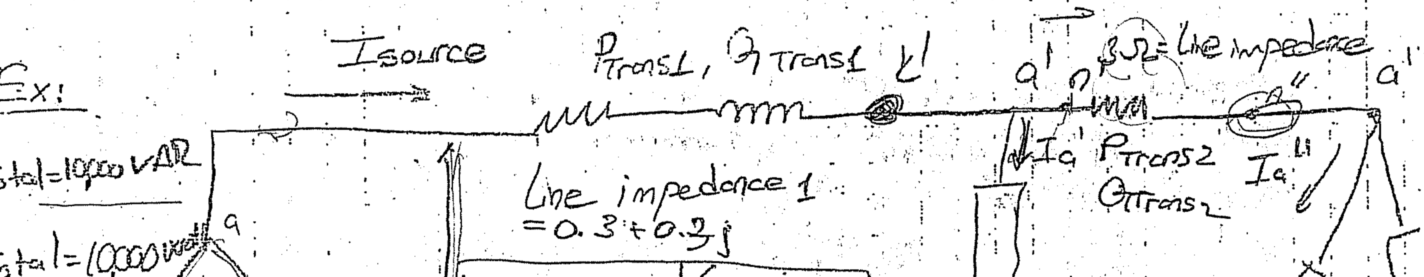
$\frac{6000^2}{3}$

$$Pf_{load\,2} = \frac{P_b}{\sqrt{P_b^2 + Q_b^2}} = \frac{2664}{2722.4} = 0.98 \text{ lagging}$$

Ex:

$Q_{Total} = 10{,}000$ VAR

$P_{Total} = 10{,}000$ Watt



Line impedance 3Ω

$V_{source_{Lin}} = \frac{\sqrt{3}}{\sqrt{2}} 200$ Volt rms

$P_{load,total} = 500$ Watt $= 9000$ Wat

$Q_{load,total} =$ VAR $= 9000$ VA

Find $|V_{a'N}|$; $|V_{a'b'}|$; $|I_{a}'|$; $|I_{a}''|$; $|V_{a''b''}|$; $|I_{n}'|$

$$S_{Total} = \sqrt{P_{Total}^2 + Q_{total}^2} = 10000\sqrt{2} \quad VA$$

$$|S_{Total}| = \sqrt{3}\,|I_{source}|\,|V_{source\,line}|$$

$$10000\sqrt{2} = \sqrt{3}\,|I_{source}|\,\frac{200\sqrt{3}}{\sqrt{2}}$$

$$|I_{source}| = \frac{100}{3}$$

$$|I_{source\,phase}| = \frac{100}{3}\cdot\frac{1}{\sqrt{3}} = \frac{100}{3\sqrt{3}} \quad A$$

$$P_{Trans\,1} = |I_{source}|^2 * 0.3 * 3 = \left(\frac{100}{3}\right)^2 * 0.3 * 3 = 1000 \,W$$

$$Q_{Trans\,1} = |I_{source}|^2 * 0.3 * 3 = \left(\frac{100}{3}\right)^2 * 0.3 * 3 = 1000 \quad VAR$$

$$P_{k'} = P_S - P_{Trans1} = 10000 - 1000 = 9000 \quad Watt$$

$$Q_{k'} = Q_S - Q_{Trans1} = 10000 - 1000 = 9000 \quad VAR$$

$$|S_{k'}| = \sqrt{P_{k'}^2 + Q_{k'}^2} = 9000\sqrt{2} \quad VA$$

$$S_{k'} = \sqrt{3}\,|I_{source}|\,|V_{k'k''}|$$

$$9000\sqrt{2} = \frac{100}{3}$$

$$9000\sqrt{2} = \sqrt{3}\,\frac{100}{3}\,|V_{k'k''}|$$

$$90\sqrt{6} = |V_{k'k''}|$$

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$$|V_{k'k''}| = |V_{a'b'}| = 90\sqrt{6} \text{ Volt}$$

$$|V_{a'N}| = \frac{|V_{a'b'}|}{\sqrt{3}} = \frac{90\sqrt{6}}{\sqrt{3}} = 90\sqrt{2} \text{ Volt}$$

$$P_n' = P_k' - P_{load1total} = 9000 - 5000 = \underline{4000} \text{ Watt}$$

$$Q_n' = Q_k' - Q_{load1total} = 9000 - 5000 = \underline{4000} \text{ VAR}$$

$$|S_n'| = \sqrt{(P_n')^2 + (Q_n')^2} = 4000\sqrt{2} \text{ VA}$$

$$|I_n'|\,|V_{k'k''}|\,\sqrt{3} = 4000\sqrt{2}$$

$$|I_n'|\,90\sqrt{6}\,\sqrt{3} = 4000\sqrt{2} \Rightarrow |I_n'| = \frac{4000}{270} = \frac{400}{27} \text{ A}$$

$$|S_{load1total}| = \sqrt{P_{load1total}^2 + Q_{load1total}^2} = 5000\sqrt{2}$$

$$|S_{load1total}| = |I_a'|\,\sqrt{3}\,|V_{a'b'}| = |I_a'| \cdot 3\,|V_{a'N}|$$

$$5000\sqrt{2} = |I_a'|\,\sqrt{3} \cdot 90\sqrt{6} \Rightarrow |I_a'| = \frac{500}{27} \text{ A}$$

$$P_{Trans\,2} = |I_n'|^2 * 3 * 3 = \left(\frac{400}{27}\right)^2 * \underline{9} = \frac{160000}{81}$$

$$Q_{Trans\,2} = 0$$

$$P_n'' = P_n' - P_{Trans\,2} = 4000 - \frac{160000}{81} = 2024.7 \text{ Watt}$$

$$Q_n'' = Q_n' - Q_{Trans\,2} = 4000 - 0 = 4000 \text{ VAR}$$

$$|S_n''| = \sqrt{(P_n'')^2 + (Q_n'')^2}$$

$$|I_a''| = \frac{I_n'}{\sqrt{3}} = \frac{400}{27} * \frac{1}{\sqrt{3}} = \frac{400}{27\sqrt{3}}$$

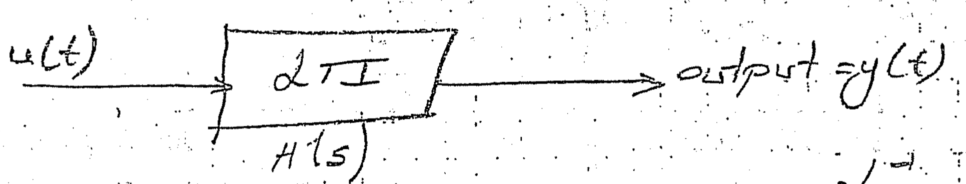
$$|S_n''| \cong 2000\sqrt{5}\ VA$$

$$|S_n''| = |I_n'|\sqrt{3}\,|V_{a''b''}|$$

$$2000\sqrt{5} = \frac{400}{27}\sqrt{3}\,|V_{a''b''}| \Rightarrow$$

$$|V_{a''b''}| = \frac{2000\sqrt{5}\cdot 27}{400\sqrt{3}} = \frac{5\sqrt{5}\,27}{\sqrt{3}}\ Volt$$

Convolution Integral:



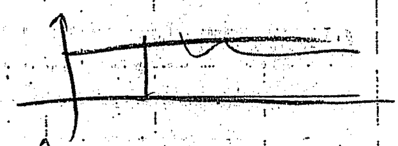
zero-state solution = $Y(s) = H(s)U(s) \xrightarrow{\mathcal{L}^{-1}} y(t) = h(t) * u(t)$

convolution

$$y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(t-\tau)\,u(\tau)\,d\tau$$

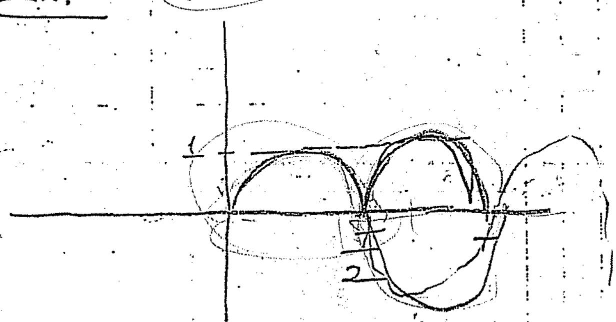
$$= \int_{-\infty}^{t} u(t-\tau)\,h(\tau)\,d\tau$$

$u(t) - u(t-1)$



Total solution = $\underbrace{\text{zero-state solution}}_{\text{convolution}}$ + add effect of initial states

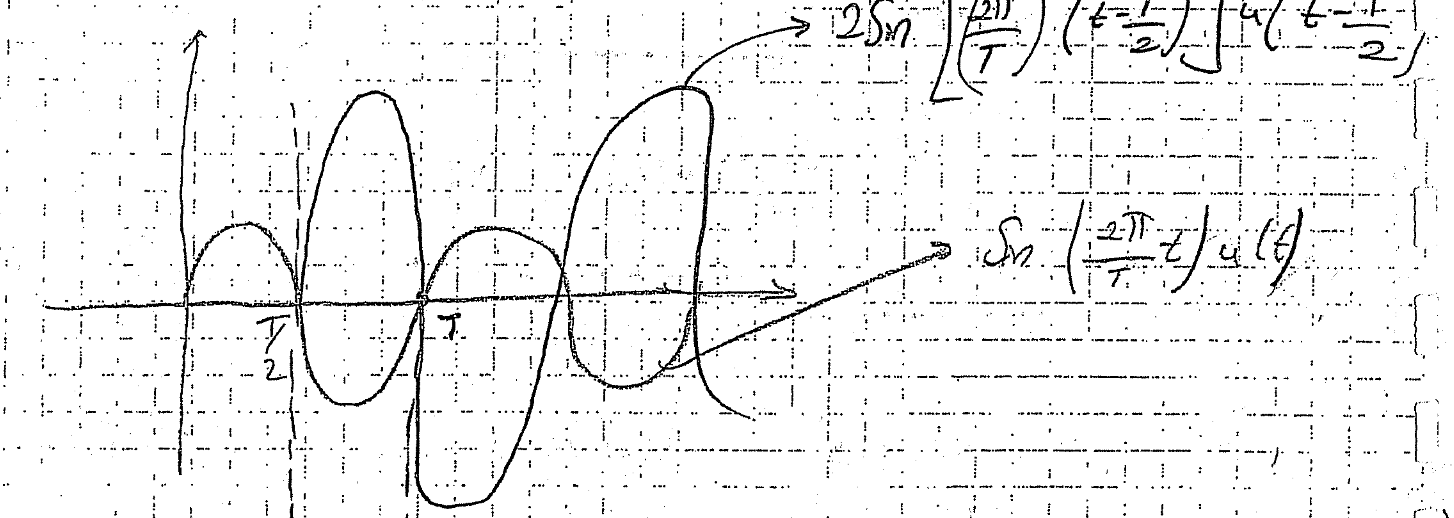
Ex: $x(t)$

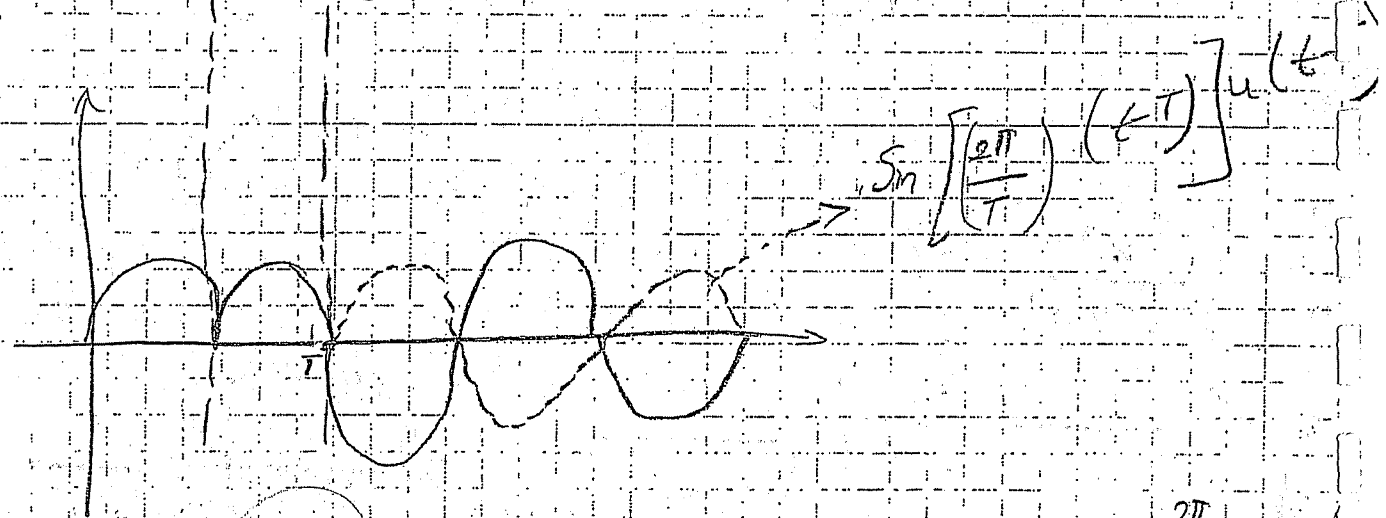


$$x(t) = \begin{cases} \sin \frac{2\pi}{T} t & 0 < t < \frac{T}{2} \\ -\sin \frac{2\pi}{T} t & \frac{T}{2} < t < \frac{2T}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \sin\left(\frac{2\pi}{T}(t-0)\right)u(t) + 2\sin\left[\left(\frac{2\pi}{T}\right)\left(t - \frac{T}{2}\right)\right]u\left(t - \frac{T}{2}\right)$$

$$+ \sin\left[\left(\frac{2\pi}{T}\right)(t - T)\right]u(t-T)$$

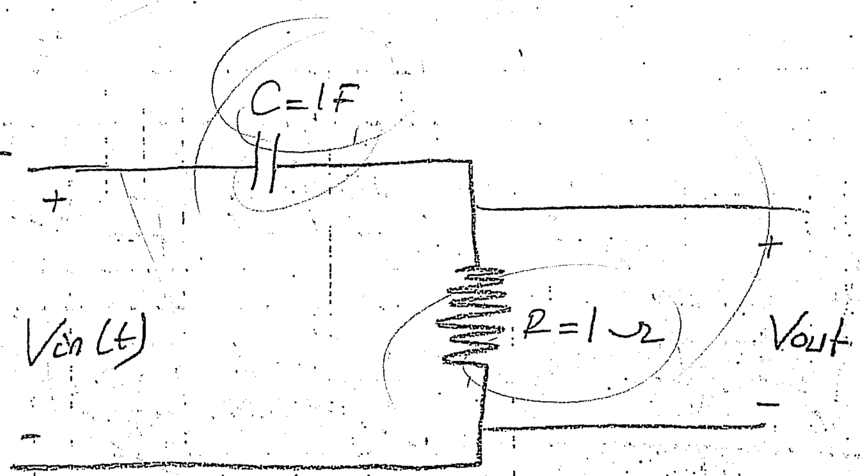




$$K(s) = \frac{\frac{2\pi}{T}}{s^2+\left(\frac{2\pi}{T}\right)^2} + 2\,\frac{\frac{2\pi}{T}}{s^2+\left(\frac{2\pi}{T}\right)^2}\,e^{-\frac{T}{2}s} + \frac{\frac{2\pi}{T}}{s^2+\left(\frac{2\pi}{T}\right)^2}\,e$$

$$K(s) = \frac{\frac{2\pi}{T}}{s^2+\left(\frac{2\pi}{T}\right)^2}\left[1+2e^{-T/2\,s}+e^{-Ts}\right]$$

Ex:

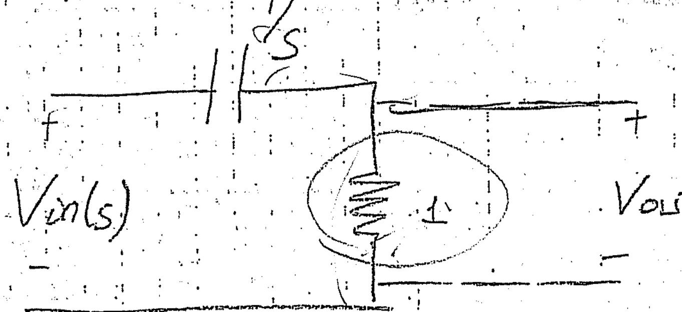


ⓐ $V_{in}(t) = e^{-t}u(t)$ What is Vout?

Use Laplace

$$V_{in}(t) = e^{-t}u(t) \xrightarrow{\mathcal{L}} V_{in}(s) = \frac{1}{s+1}$$

$$\frac{V_{out}(s)}{V_{in}(s)} = ?$$



$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1+\frac{1}{s}} = \frac{s}{s+1}$$

$$V_{out}(s) = V_{in}(s)\,H(s) = \frac{1}{s+1}\cdot\frac{s}{s+1} = \frac{s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$As + A + B = s$$

$$A=1 \quad A+B=0 \Rightarrow B=-1$$

$$V_{out}(s) = \boxed{\frac{1}{s+1} + \frac{-1}{(s+1)^2}} \xrightarrow{\mathcal{L}^{-1}} V_{out}(t) = e^{-t}u(t) - te^{-t}u(t)$$

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(b) $V_{in}(t) = e^{-t}u(t)$ $\quad$ $V_{out}(t) = ?$ $\rightarrow$ use convolution

$$V_{out}(t) = \int_{-\infty}^{t} V_{in}(t-\tau)\,h(\tau)\,d\tau$$

$$H(s) \xrightarrow{\mathcal{L}^{-1}} h(t)$$

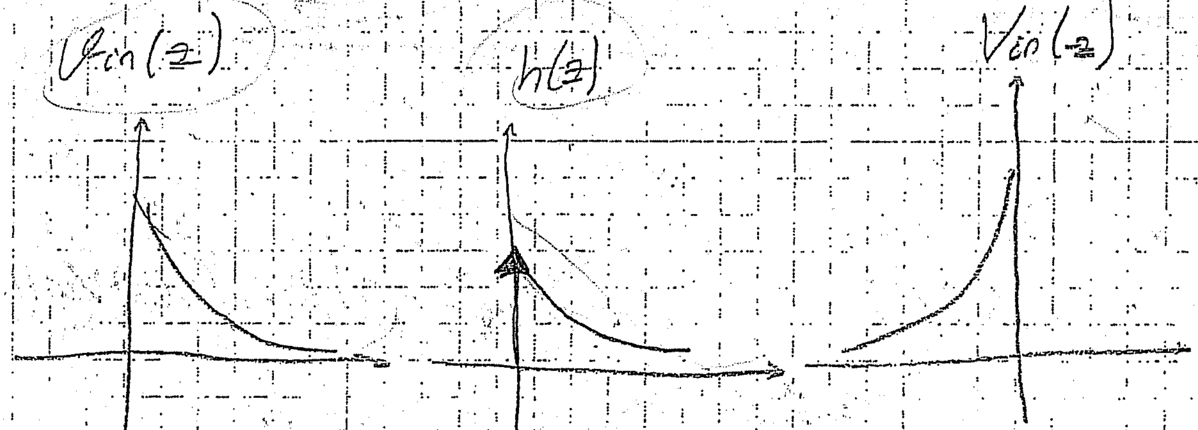
$H(s)$: transfer function $\qquad$ $h(t)$: impulse response

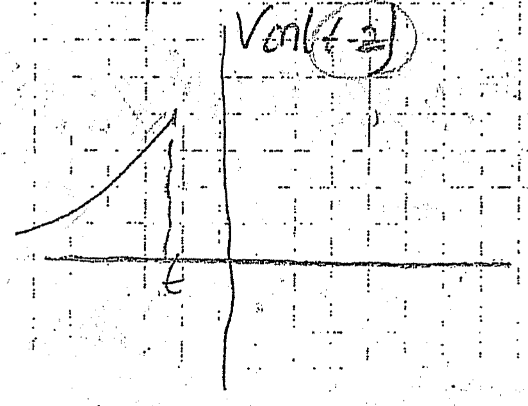
$$H(s) = \frac{s}{s+1} = \frac{s+1-1}{s+1} = 1 - \frac{1}{s+1}$$

$\mathcal{L}^{-1}$

$$h(t) = \delta(t) - e^{-t}u(t)$$

$$V_{in}(t) = e^{-t}u(t)$$





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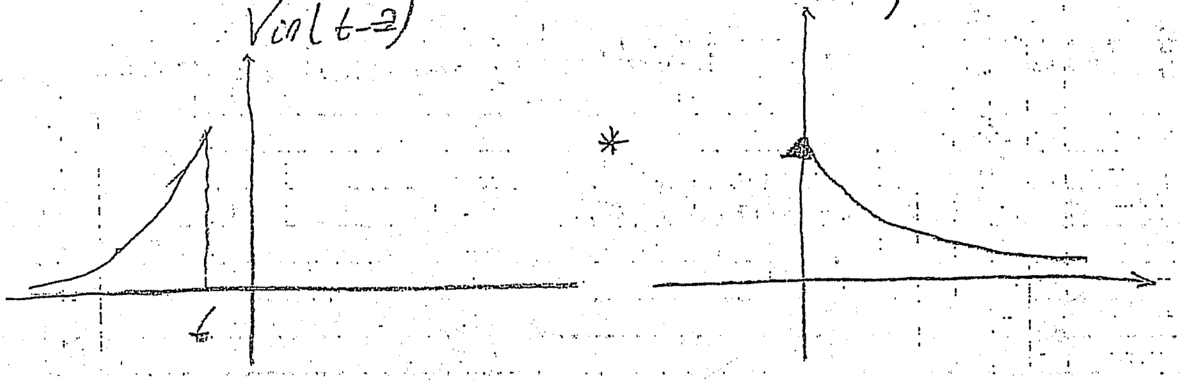
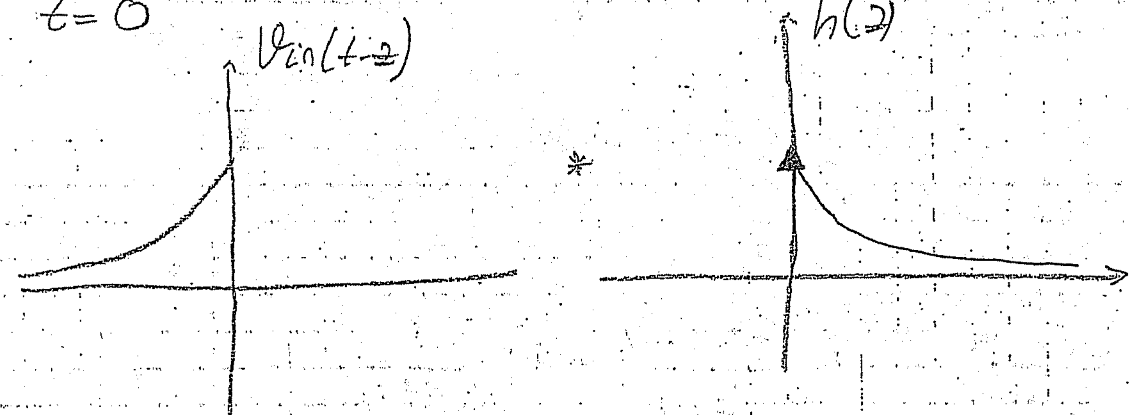
$t<0$



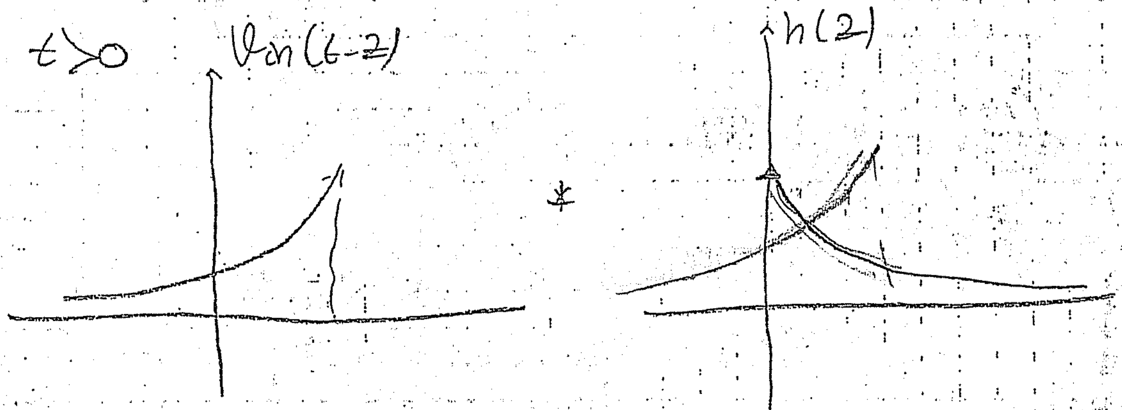
$=0$

$t=0$



$\neq 0$

$t>0$



$\neq 0$

$t<0 \qquad V_{out}(t) = 0$

$$t=0 \qquad V_{out}(t) = \cancel{\int_{-\infty}^{0^-} h(\tau)\, V_{in}(t-\tau)\, d\tau} + \int_{0^-}^{0^+} h(\tau)\, V_{in}(t-\tau)\, d\tau$$

$$= 0 + \int_{0^-}^{0^+} \left[\delta(\tau)\, e^{-\tau} u(\tau) \right] e^{-(t-\tau)} u(t-\tau)\, d\tau$$

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$$V_{out}(t) = 0 + e^{-t}\left[\int_{0^-}^{0^+}\left[\delta(z) - e^{-z}u(z)\right]e^{z}u(t-z)\,dz\right]$$

$$V_{out}(t) = e^{-t}\left[\int_{0^-}^{0^+}\delta(z)e^{z}u(t-z)\,dz - \underbrace{\int_{0^-}^{0^+} 1\, u(z)\,u(t-z)\,dz}_{0}\right]$$

$$V_{out}(t) = e^{-t}\left[\int_{0^-}^{0^+}\delta(z)\underbrace{e^{z}}_{1}\underbrace{u(t-z)}_{1}\,dz\right]$$

$$\boxed{\delta(z)f(z) = \delta(z)f(0)}$$

$t=0$

$V_{out}(t) = e^{-t}$ $\quad V_{out}(0) = e^{-0} = 1$

$t>0$

$$V_{out}(t) = \underbrace{\int_{-\infty}^{0^-} h(z)V_{in}(t-z)\,dz}_{0} + \int_{0^-}^{0^+} h(z)V_{in}(t-z)\,dz + \int_{0^+}^{t} h(z)V_{in}$$

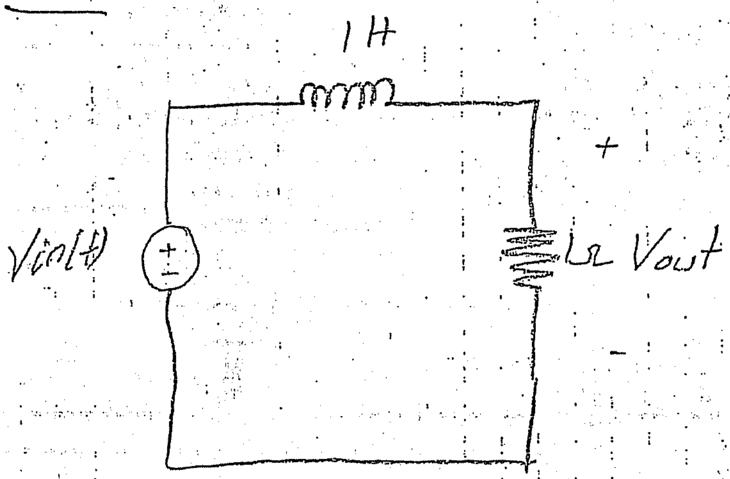
$$V_{out}(t) = 0 + e^{-t} + \int_{0^+}^{t}\left[\delta(z) - e^{-z}u(z)\right]e^{-(t-z)}u(t$$

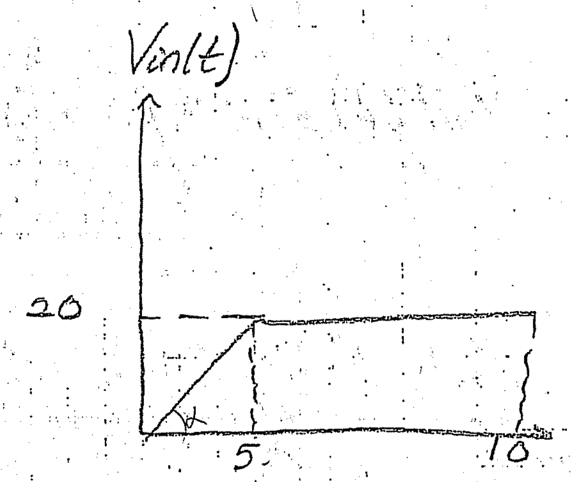
$$V_{out}(t) = e^{-t} + \int_{0^+}^{t}\delta(z)e^{-(t-z)}u(t-z)\,dz - \int_{0^+}^{t} e^{-t}u(z)\,u(t-z)\,dz$$

$$= 0 - \int_{0^+}^{t} e^{-t}u(z)u(t-z)\,dz = -\int_{0^+}^{t} e^{-t}\,dz + e$$

$$V_{out}(t) = e^{-t} - e^{-t}\int_{0^+}^{t} 1 \, d\tau = e^{-t} - e^{-t} t$$

Ex:



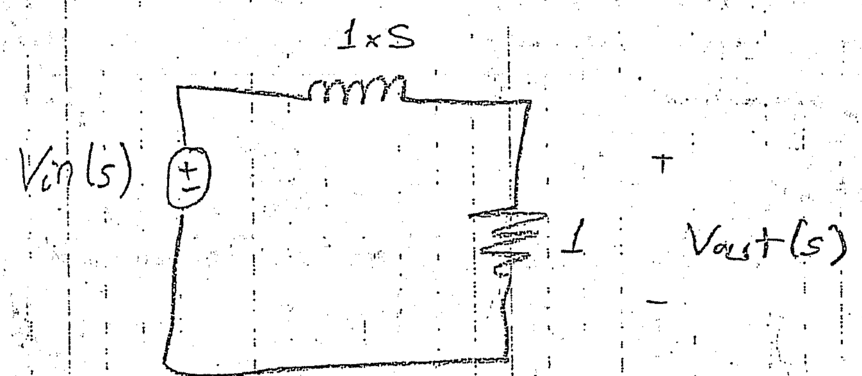


Find $V_{out}(t) = ?$

Solution:

$$V_{out}(t) = \int_{-\infty}^{t} h(\tau)\, V_{in}(t-\tau)\, d\tau = \int_{-\infty}^{t} V_{in}(\tau)\, h(t-\tau)\, d\tau$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{s+1} = \text{Transfer function}$$



$$H(s) = \frac{1}{S+1} \xrightarrow{\mathcal{L}^{-1}} h(t) = e^{-t} u(t)$$

impulse response

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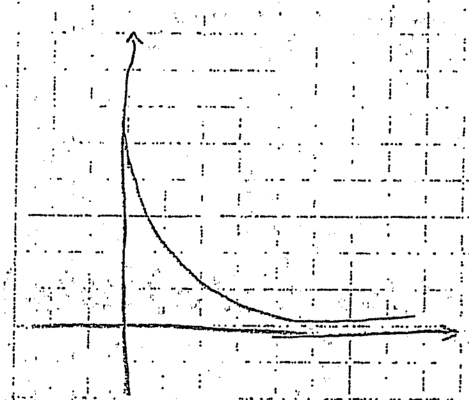
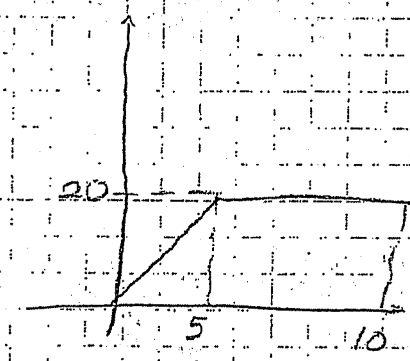
$$V_{in}(t) = \begin{cases} 0 & -\infty < t < 0 \\ 4t & 0 \le t < 5 \\ 20 & 5 \le t < 10 \\ 0 & t > 10 \end{cases}$$

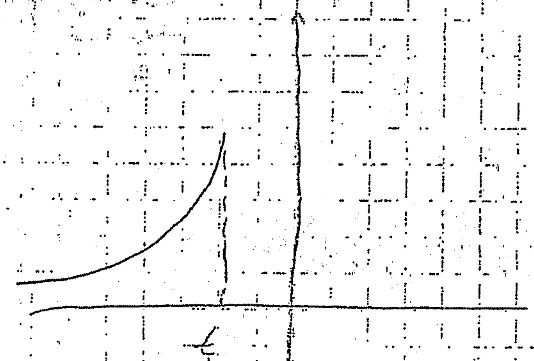
$h(\tau)$



$V_{in}(\tau)$



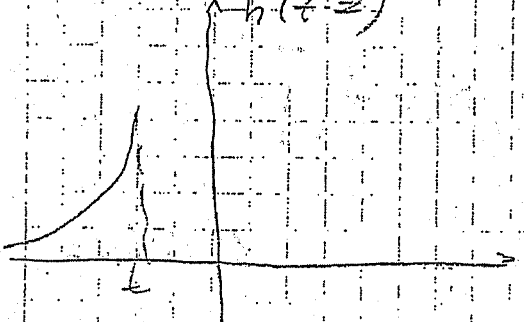
$$h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$$



t → parameter

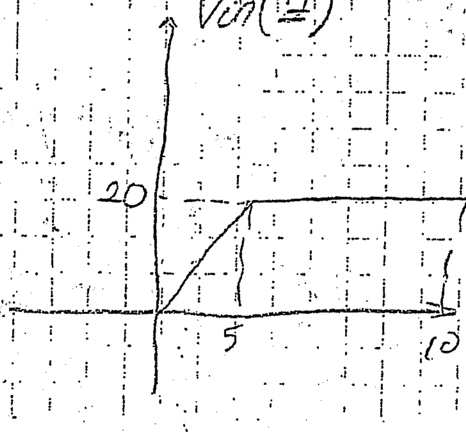
If $t < 0$

$h(t-\tau)$



$*$

$V_{in}(\tau)$



$= 0$

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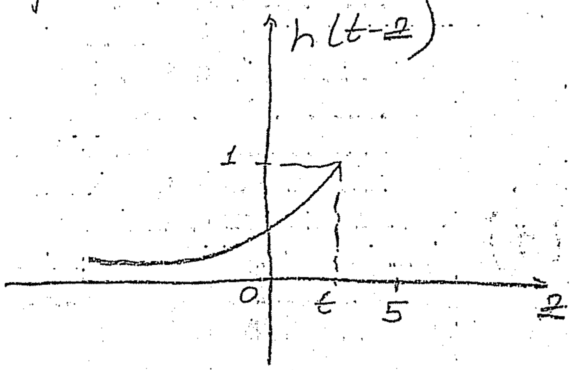
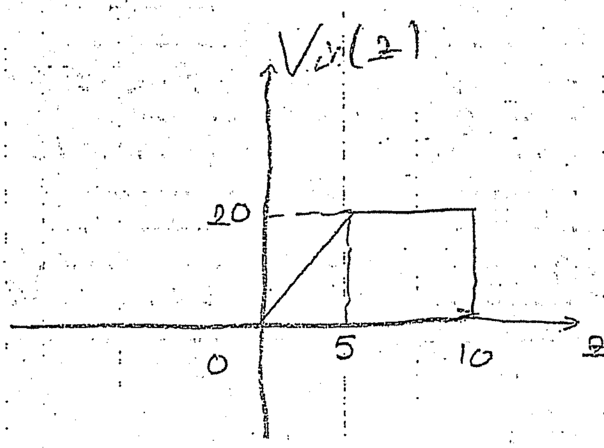
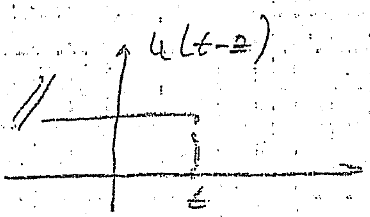
$$\int_{-\infty}^{t} h(t-z)\, V_{in}(z)\, dz = 0$$

if $0 < t < 5$





$$\int_{-\infty}^{t} h(t-z)\, V_{in}(z)\,dz = \underbrace{\int_{-\infty}^{0} h(t-z)\, \underbrace{V_{in}(z)}_{0}\,dz}_{0} + \int_{0}^{t} h(t-z)\, V_{in}(z)\,dz$$



$$+ \int_{0}^{t} e^{-(t-z)}\, \underbrace{u(t-z)}_{1}\, 4z\, dz$$

$$= 0 + \int_{0}^{t} e^{-t} e^{z}\, 4z\, dz = 4e^{-t} \int_{0}^{t} e^{z}\, z\, dz$$

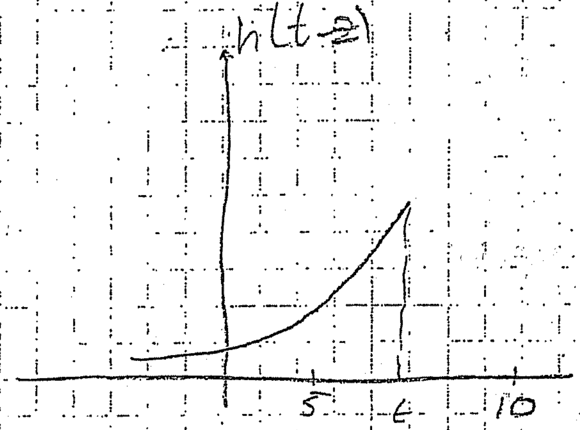
$$\begin{array}{cc} z & e^{z} \\ 1 & e^{z} \\ 0 & e^{z} \end{array}$$

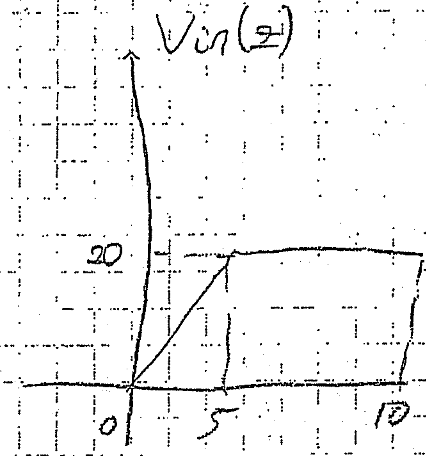
$$= 4e^{-t}\left[z e^{z} - e^{z} \right]_{0}^{t}$$

$$= 4e^{-t}\left[t e^{t} - e^{t} + 1 \right]$$

$$V_{out}(t) = 4t - 4 + 4e^{-t} \qquad 0 \le t < 5$$

if $5 < t < 10$





$$V_{out}(t) = \int_{-\infty}^{0} h(t-z)\,V_{in}(z)\,dz + \int_{0}^{5} h(t-z)\,V_{in}(z)\,dz + \int_{5}^{t} h(t-z)\,V_{in}(z)\,dz$$

$$= 0 + \int_{0}^{5} e^{-(t-z)} \underbrace{u(t-z)}_{1}\, 4z\,dz + \int_{5}^{t} e^{-(t-z)}\, u(t-z)\, 20\,dz$$

$$= e^{-t}\int_{0}^{5} e^{z}\, 4z\,dz + e^{-t}\int_{5}^{t} e^{z}\, 20\,dz$$

$$= e^{-t}\left[4\cdot 5\cdot e^{5} - 4e^{5} + 4\right] + 20e^{-t}\left(e^{t} - e^{5}\right)$$

$$= e^{-t}\left[16e^{5} + 4\right] + 20 - 20e^{5-t}$$

$$= e^{-t}\left[16e^{5} + 4 + 20e^{t} - 20e^{5}\right]$$

$$= e^{-t}\left[-4e^{5} + 4\right] + 20$$

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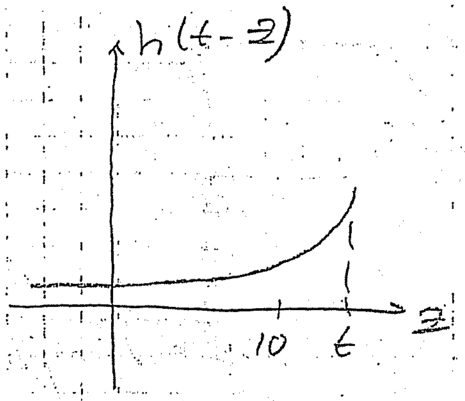
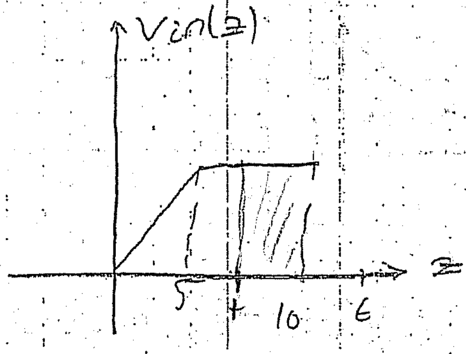
$$V_{out}(t) = e^{-t}\left[4-4e^{5}\right]+20 \qquad 5<t\leq 10$$

if $t>10$





$$V_{out}(t) = \int_{-\infty}^{t} h(t-z)\,V_{in}(z)\,dz = \underbrace{\int_{-\infty}^{0} ---}_{0} + \underbrace{\int_{0}^{5} ---} + \int_{5}^{10} --- + \underbrace{\int_{10}^{t} ---}_{0}$$

$$V_{out}(t) = e^{-t}\left[16e^{5}+4\right] + \int_{5}^{10} e^{-(t-z)}\,\underbrace{u(t-z)}_{1}\,20\,dz$$

$$V_{out}(t) = e^{-t}\left[16e^{5}+4\right] + e^{-t}\int_{5}^{10} e^{z}\,20\,dz$$

$$= e^{-t}\left[16e^{5}+4\right] + 20e^{-t}\left[e^{10}-e^{5}\right]$$

$$V_{out}(t) = e^{-t}\left[-4e^{5}+4+20e^{10}\right] \qquad t>10$$

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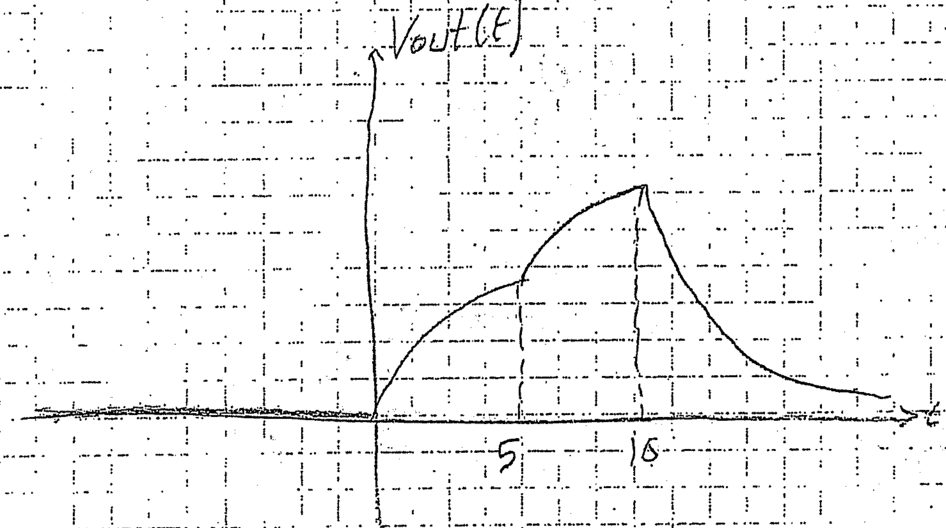


$$V_{out}(t) = \begin{cases} 0 & t < 0 \\ 4t - 4 + 4e^{-t} & 0 < t < 5 \\ e^{-t}\left[4 - 4e^{5}\right] + 20 & 5 < t < 10 \\ e^{-t}\left[4 - 4e^{5} + 20e^{10}\right] & 10 < t \end{cases}$$

$V_{out}(0) = 0$

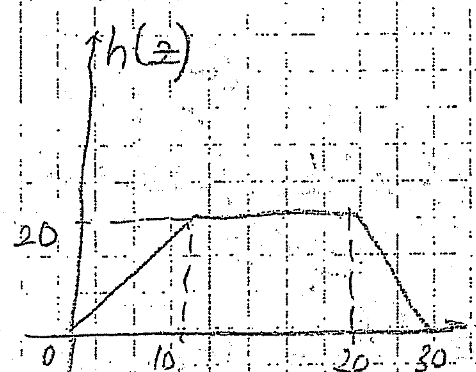
$V_{out}(5) = 16 + 4e^{-5}$

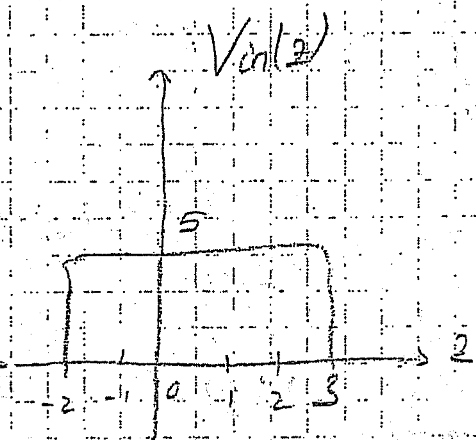
$V_{out}(5) = 16 + 4e^{-5}$

$V_{out}(10) = 4e^{-10} - 4e^{-5} + \dots$

$V_{out}(10) = 4e^{-10} - 4e^{-5} + 20$

Ex:





$V_{out}(t) = h(t) * V_{in}(t)$

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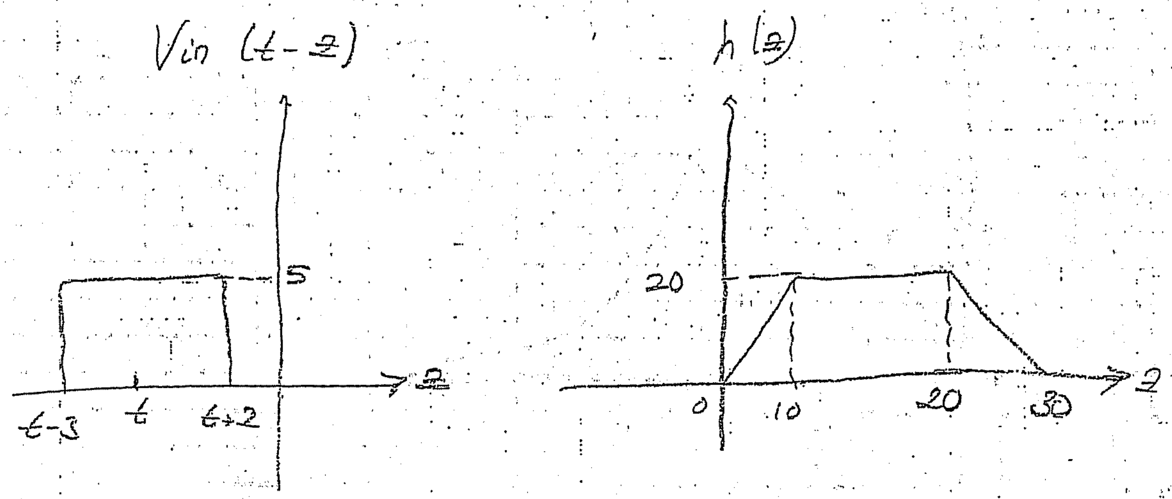
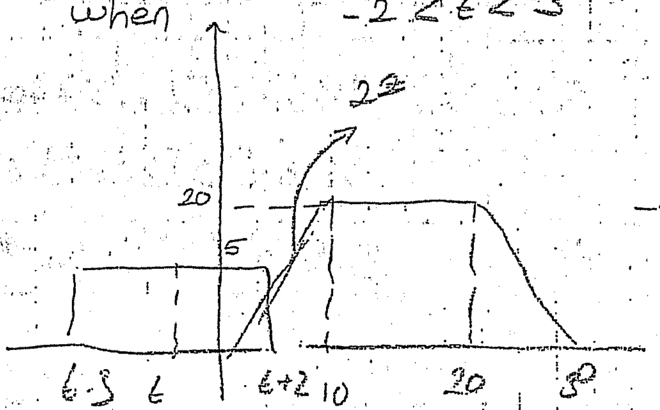
Solution: when $-\infty < t < -2$



$$V_{out}(t) = \int_{-\infty}^{t} h(z)\, V_{in}(t-z)\, dz = 0$$

when $-2 < t < 3$



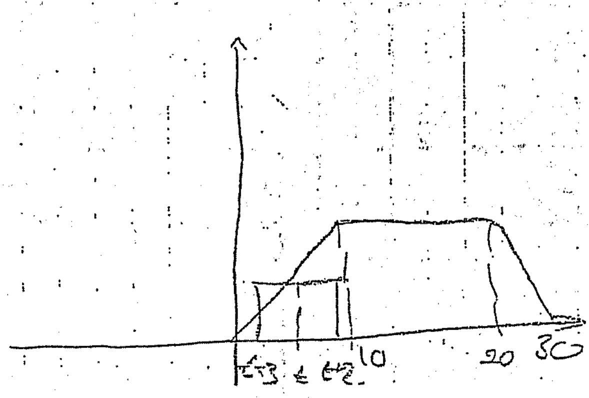
$$\int_{-\infty}^{t+2} h(z)\, V_{in}(t-z)\, dz = \underbrace{\int_{-\infty}^{0} \cdots}_{0} + \int_{0}^{t+2} 2z \cdot 5\, dz$$

$$V_{out}(t) = 5z^2 \Big|_{0}^{t+2} = 5(t+2)^2$$

when $3 < t < 8$

$$\int_{-\infty}^{t+2} h(z)\, V_{in}(t-z)\, dz = \int_{t-3}^{t+2} 2z \cdot 5\, dz$$

$$V_{out}(t) = 5z^2 \Big|_{t-3}^{t+2} = 50t - 25$$



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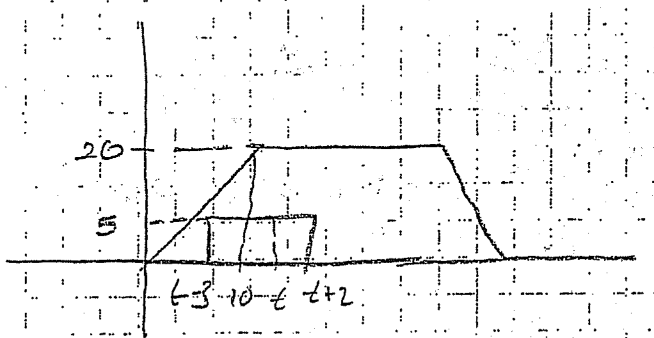
$8 < t < 13$



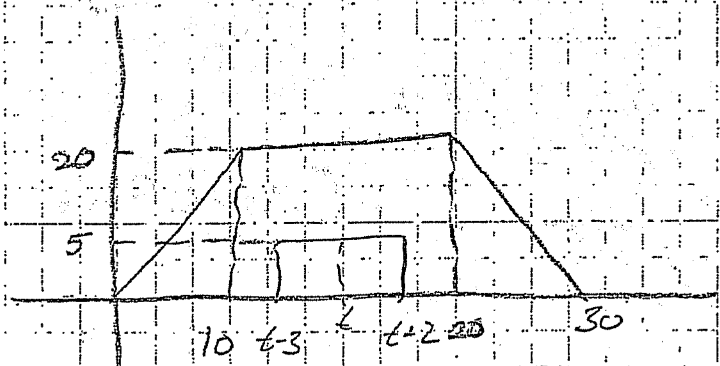
$$\int_{-\infty}^{t+2} \text{---} = 0 + \int_{t-3}^{10} 5\,2z\,dz + \int_{10}^{t+2} 20 \times 5\,dz$$

$$V_{out}(t) = 5z^2 \Big|_{t-3}^{10} + 100\,z \Big|_{10}^{t+2}$$

$$= 500 - 5t^2 + 30t - 45 + 100t - 800$$

$$= -345 + 130t - 5t^2$$

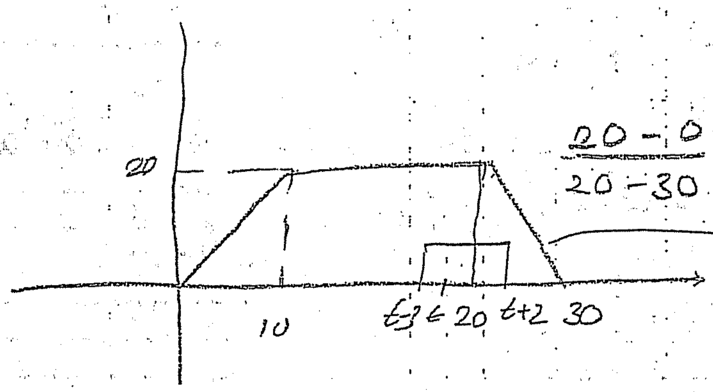
$13 < t < 18$



$$\int_{-\infty}^{t+2} \text{---} = \int_{t-3}^{t+2} 5.20\,dz = 100\,z \Big|_{t-3}^{t+2} = 500$$

203 15 05 07

$18 < t < 23$



$$\frac{20-0}{20-30} = \frac{y-0}{z-30} \qquad -2 = \frac{y}{z-30}$$

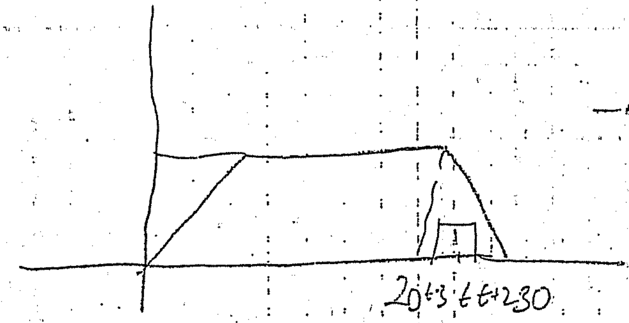
$$y = -2z + 60$$

$$\int_{-\infty}^{t+2} \text{------} = \int_{t-3}^{20} 20 \cdot 5 \, dz + \int_{20}^{t+2} 5 \cdot (-2z + 60) \, dz$$

$$= 100 z \Big|_{t-3}^{20} + \Big[-5z^2 + 300 z \Big|_{20}^{t+2}$$

$$= 100 \left[20 - t + 3 \right] - 5 \left[(t+2)^2 - 20^2 \right] + 300 \left[t + 2 - 20 \right]$$

$23 < t < 28$



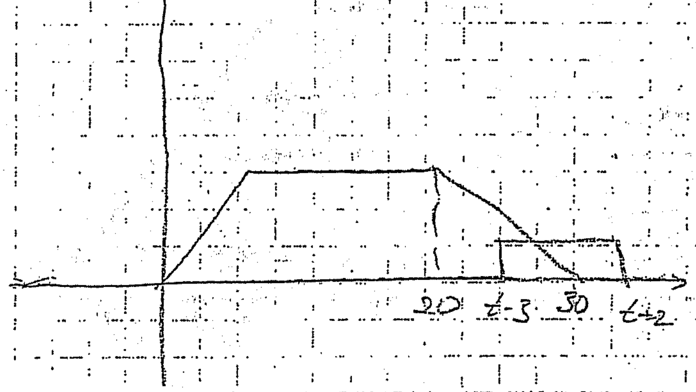
$$\int_{-\infty}^{t+2} \text{----} = \int_{t-3}^{t+2} 5 \left[-2z + 60 \right] dz$$

$$= -5z^2 + 300 z \Big|_{t-3}^{t+2}$$

$$V_{out}(t) = -5 \left[(t+2)^2 - (t-3)^2 \right] + 300 \left[t + 2 - t + 3 \right]$$

$$= -5 \left[(2t-1) 5 \right] + 300 \cdot 5 = -5 \left[10t - 5 \right] + 1500$$

$28 < t < 33$

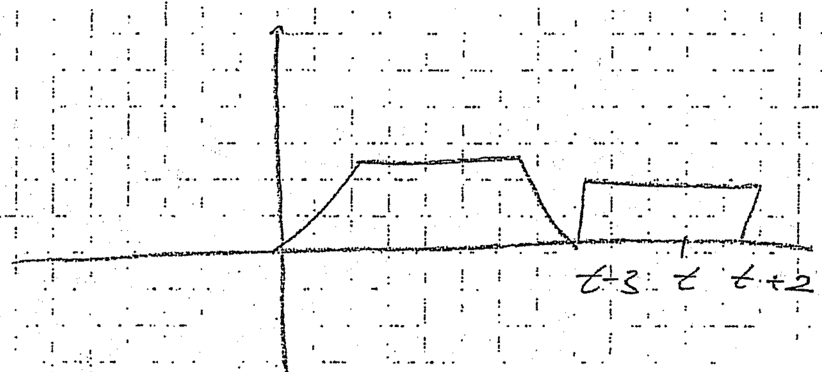


$$\int_{-\infty}^{t+2} ---- = \int_{t-3}^{30} 5\left[-2\tau+60\right] = -5\tau^2+300\tau \Big|_{t-3}^{30}$$

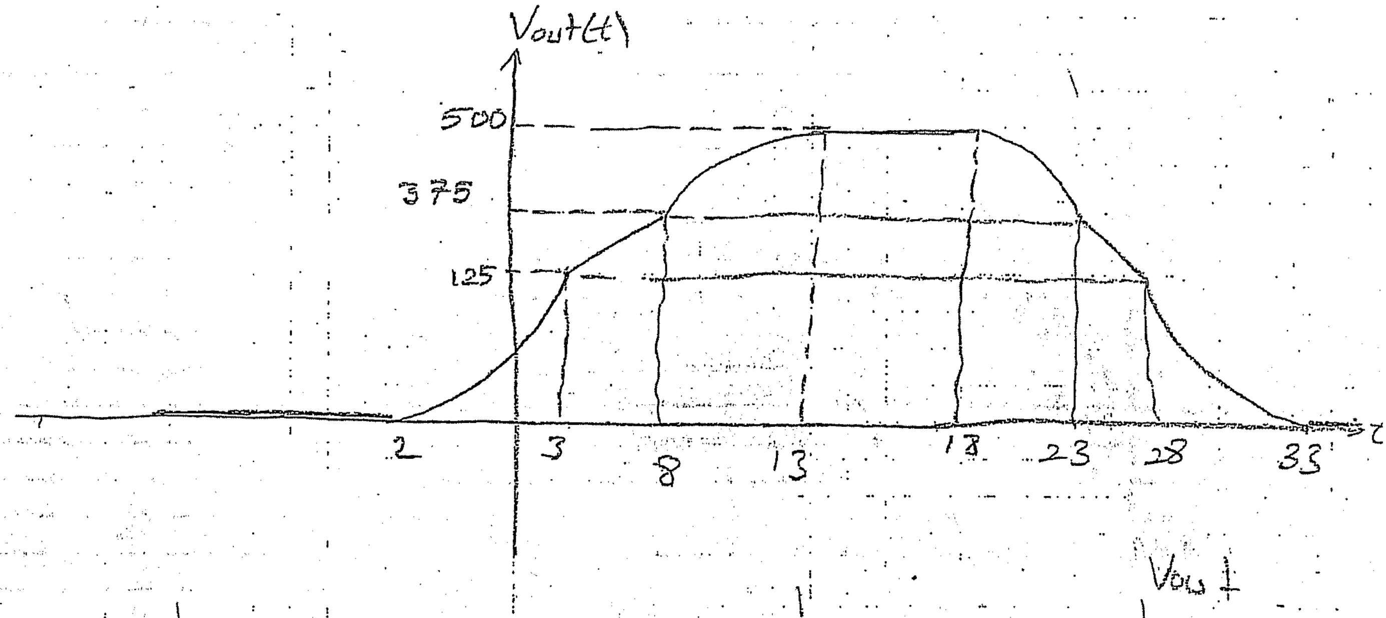
$$= -5\left[900-t^2+6t-9\right]+300\left[33-t\right]$$

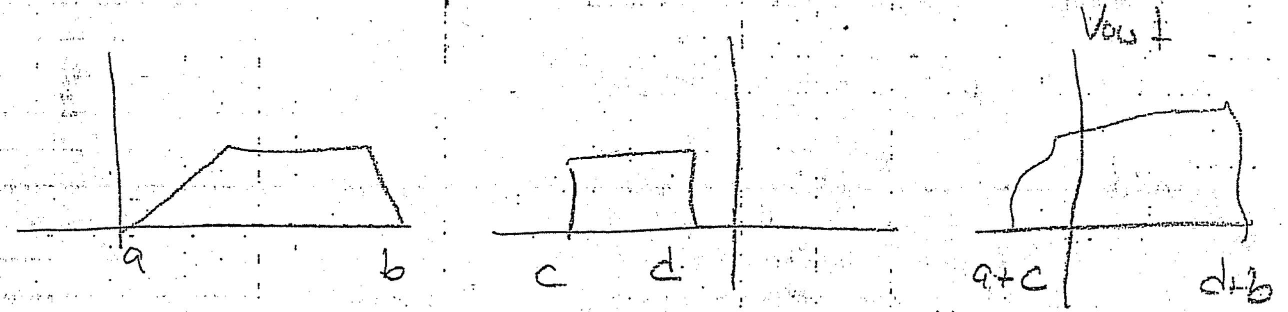
$33 < t$

$V_{out}(t) = 0$



$$V_{out}(t) = \begin{cases} 0 & t < -2 \\ 5(t+2)^2 & -2 < t < 3 \\ 50t-25 & 3 < t < 8 \\ -5t^2+130t-345 & 8 < t < 13 \\ 560 & 13 < t < 18 \\ -5t^2-180t-1120 & 18 < t < 23 \end{cases}$$


Vout(t)
500
375
125
2
3
8
13
18
23
28
33
t


a
b
c
d
Vout
a+c
d+b