AC Circuits:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

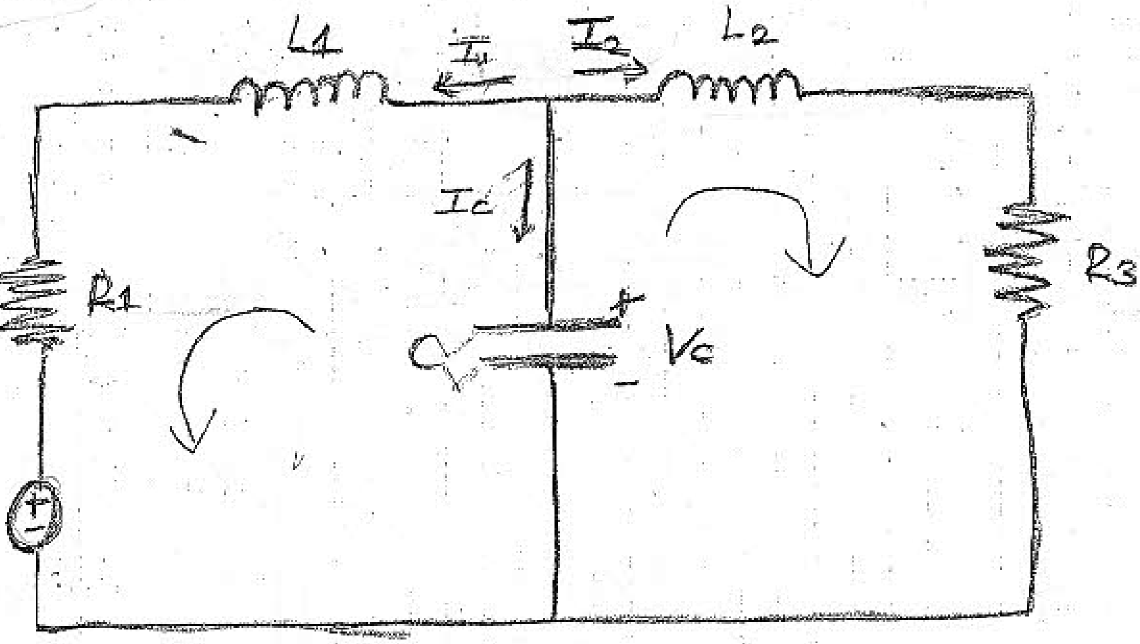
state space representation

derivatives of state variables

state variables

input vector (source vector)

Ex:



V_c ; I_1 ; I_2 ⟷ states

$$-I_1 - I_2 = I_c$$

$$I_1 + I_2 + C\frac{dV_c}{dt} = 0$$

$$I_1 + I_2 + I_c = 0$$

$$I_1 + I_2 + C\frac{dV_c}{dt} = 0$$

$$V_{L_1} + V_{R_1} + V_S = V_C$$

$$V_{L_2} + V_{R_3} = V_C$$

$$\Longrightarrow \boxed{\frac{dV_c}{dt} = \frac{-I_1}{C} - \frac{I_2}{C}}$$

$$L_1\frac{dI_1}{dt} + V_{R_1} + V_S = V_C$$

$$L_2\frac{dI_2}{dt} + V_{R_3} = V_C$$

$$\frac{dI_2}{dt} = \frac{V_c}{L_2} - \frac{V_{R3}}{L_2}$$

$$V_{L_1} + V_{R_1} + V_S = V_C$$

$$L_1 \frac{dI_1}{dt} + R_1 I_1 + V_S = V_C$$

$$\Longrightarrow \boxed{\frac{dI_1}{dt} = \frac{-V_S}{L_1} - \frac{R_1}{L_1} I_1 + \frac{V_C}{L_1}}$$

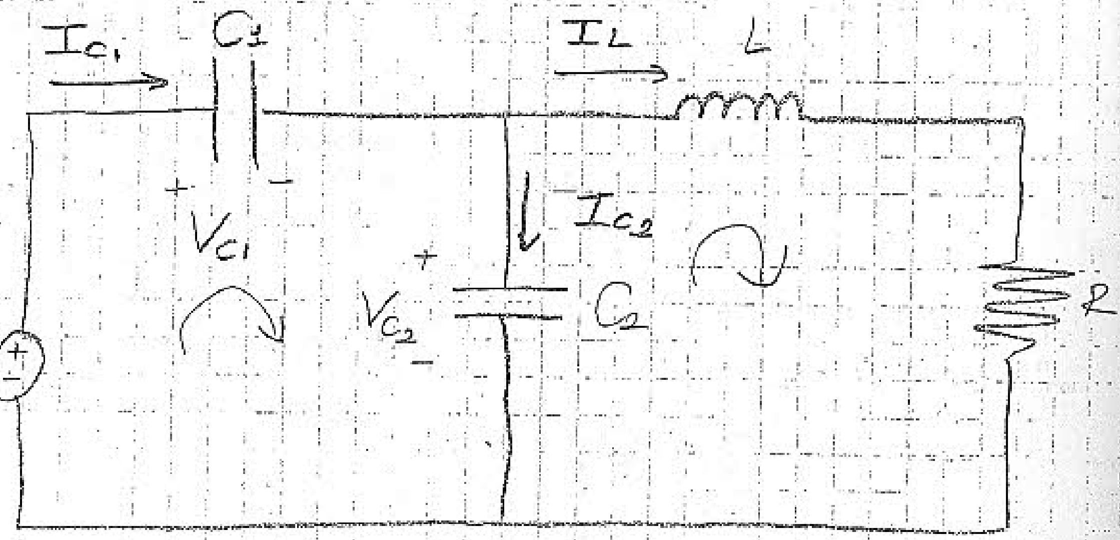
$$V_C = V_{L_2} + V_{R_3}$$

$$V_C = L_2 \frac{dI_2}{dt} + I_2 . R_3$$

$$\Longrightarrow \boxed{\frac{dI_2}{dt} = \frac{V_C}{L_2} - \frac{R_3}{L_2} I_2}$$

$$? \begin{bmatrix} \dot{V_C} \\ \dot{I_1} \\ \dot{I_2} \end{bmatrix} = \begin{bmatrix} 0 & -1/C & -1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ 1/L_2 & 0 & -R_3/L_2 \end{bmatrix} \begin{bmatrix} V_C \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} V_s$$

Ex:



$$I_{C1} = I_{C2} + I_L$$

$$C_1 \frac{dV_{C1}}{dt} = C_2 \frac{dV_{C2}}{dt} + I_L$$

$$\boxed{V_S = V_{C1} + V_{C2}}$$

$$I_{C1} = C_1 \frac{dV_{C1}}{dt} \Longrightarrow I_{C1} = C_1 \frac{d}{dt}\left[V_S - V_{C2}\right]$$

$$C_2 \frac{dV_{C2}}{dt} + I_L = C_1 \frac{d}{dt}\left[V_S - V_{C2}\right]$$

$$(C_1 + C_2) \frac{dV_{C2}}{dt} = C_1 \frac{dV_S}{dt} - I_L$$

$$\boxed{\frac{dV_{C2}}{dt} = \frac{C_1}{C_1 + C_2} \frac{dV_S}{dt} - \frac{1}{C_1 + C_2} I_L}$$

$$V_{C2} = V_L + V_R$$

$$V_{C2} = L \frac{dI_L}{dt} + R \cdot I_L \Longrightarrow$$

$$\boxed{\frac{dI_L}{dt} = \frac{V_{C2}}{L} - \frac{R}{L} I_L}$$

?

$$\begin{bmatrix} \dot{V}_{C2} \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} 0 & -1/(C_1 + C_2) \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} V_{C2} \\ I_L \end{bmatrix} + \begin{bmatrix} C_1/(C_1 + C_2) \\ 0 \end{bmatrix} \frac{dV_S}{dt}$$

$$\boxed{I_{C1} - I_{C2} = I_L}$$

$$-V_S + V_{C1} + V_{C2} = 0$$

$$V_L + V_R = V_{C2}$$

* Either V_{C1} or V_{C2} can be written in terms of the other variable without any state equation using $\boxed{-V_S + V_{C1} + V_{C2} = 0}$

13.02.07

Analysis of N^{th} order circuits:

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_n x = b_0 \frac{d^m u(t)}{dt^m} + b_1 \frac{d^{m-1} u(t)}{dt^{m-1}} + \dots + b_m u$$

requires N initial conditions

$$x(0);\ \frac{dx}{dt}(0);\ \dots\dots;\ \frac{d^{n-1}x(0)}{dt^{n-1}}$$

$$x(t) = X_{\text{zero-state}}(t) + X_{\text{zero-input}}(t)$$

due to input only $(u(t))$ and initial conditions are zero $\left(x(0) = 0; \frac{dx}{dt}(0) = 0; \dots; \frac{d^{n-1}x}{dt^{n-1}}(0) = 0 \right)$

due to initial conditions (initial conditions are non-zero) but $u(t) = 0$

$x(t) \longrightarrow$ state

$u(t) \longrightarrow$ input

Let's find zero-input solution

assume $u(t) = 0$; instead of "$\frac{d}{dt}$" terms put "s"

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$$

$\hookrightarrow$ characteristic equation

* Zeros (roots) of the characteristic equation are the natural frequencies of the circuit.

* If all the natural frequencies of the circuit are distinct (different from each other)

$$X_{homogeneous} = X_{zero-input} = \sum_{i=1}^{n} K_i \, e^{s_i t}$$

$n \rightarrow$ order of chac. eqn.

(not always true)

$\rightarrow$ parameters unknown due to initial condition

$\rightarrow$ parameters are evaluated

$s_i \neq s_k$ for all $1 < i < n$, $1 < k < n$

$\boxed{s_i = \text{natural frequency}}$

$K_i =$ constant to be determined by initial conditions

* If there are some natural frequencies with are equal to each other. (for some i and k $s_i = s_k$)

$\rightarrow$ meaning some natural frequency has a multiplicity.

(Ex: $s_1 = -2$, $s_2 = -2$, $s_4 = -2$
$s_1 = s_2 = s_4$ multiplicity of $s = -2$ is 3)

$$X_h(t) = \sum_{i=1}^{N} \sum_{m=0}^{d_k - 1} K_{im} \, t^m \, e^{s_k t}$$

$N = \#$ of distinct roots

$\downarrow$
number

s_k = natural frequencies with multiplicities d_k

Ex: $(s-1)(s-1)(s-2)(s+j)(s-j)(s+j)(s-j) = 0$

$s=1$ $s=2$ $-j$ j $-j$ j

7 roots

4 distinct roots

multiplicity of $s_1 = 1$ 2

" " $s_2 = 2$ 1

" " $s_3 = -j$ 2

" " $s_4 = j$ 2

$= K_{10}\, t e^{t} + K_{11} e^{t} + K_{20} e^{2t} + K_{30}\, t e^{-jt} + K_{31} e$

$+ K_{40}\, t e^{jt} + K_{41} e^{jt}$

Complete Response:

$\boxed{X(t) = X_h(t) + X_p(t)}$ (generally $X_p(t)$ (particular solution) is similar to input $u(t)$)

Let $\underbrace{u(t)}_{\text{input}} = \underbrace{V_0}_{\text{amplitude}} \cos(\omega t + \theta)$

$\theta \rightarrow$ phase

$\omega \rightarrow$ angular frequency

 13.02.07

what is $X_p(t)$?

↳ 2 situation

↳ ① $\boxed{s_k \neq j\omega}$ for $k = 1, ---, N$

s_k → any natural frequency

$j\omega$ → angular frequency of input

$$X_p(t) = A_m \cos(\omega t + \theta + \phi)$$

A_m → amplitude will change

ω → angular frequency is same

ϕ → extra phase difference

↳ ② $s_k = j\omega$ for some k (no real part for s_k exist)

s_k is purely imagınary
$s_k = 0 + jb$

$$X_p(t) = t^{d_k} A_m \cos(\omega t + \theta + \phi)$$

d_k is the multiplicity of s_k.

Characteristic of complete solution (giving same input $u(t) = V_o \cos(\omega t + \theta)$)

① K_{im} values are determined according to initial conditions such that $X_h(t) = 0$ in that case

$$X(t) = \underbrace{X_h(t)}_{0} + X_p(t)$$

$$X(t) = X_p(t) \quad \left(\text{since } x_h(t) = 0 \text{ there will be no trasient solution.} \right)$$

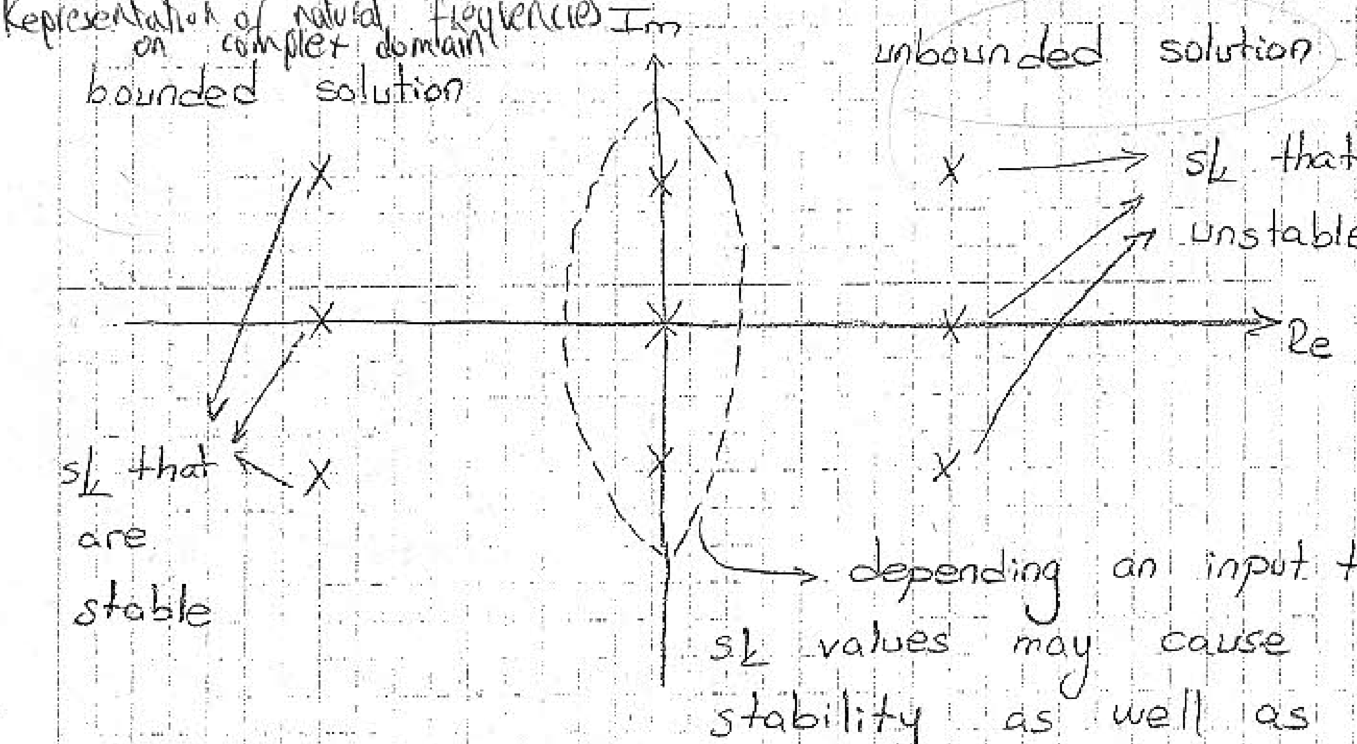
② $Re\{s_k\} < 0 \qquad k = 1, \ldots, N$

as $t \longrightarrow \infty$, $\left| s^{kt} \right| \longrightarrow 0$; homogeneous solut will go to 0 as $t \longrightarrow \infty$ and only particula solution will remain. (Steady-state) (SS)
↳ $X_p(t)$

Special case $\left(u(t) = V_0 \cos(\omega t + \theta) \right)$

as $t \longrightarrow \infty$ $\quad x(t) = X_p(t) \longrightarrow$ sinusoidal stea state solution is obtained (SSS)

Representation of natural frequencies on complex domain



→ depending an input the sk values may cause stability as well as instability.

13 02 07

(3) $Re\{s_k\} \le 0 \quad k=1,\dots,N$ } same roots (zeros) have zero real parts } and $u(t) = V_0 \cos(\omega t + \theta)$

same input

a) $s_k \neq j\omega$

let $x_{h_0}(t)$ be a linear combination of sinusoidals due to roots with zero real part.

$$x(t) = x_h(t) + x_p(t)$$

$$x_h(t) = \underbrace{x_{h_0}(t)}_{\text{some sinusoidals + some DC terms}} + \underbrace{x_{h_{non\text{-}0}}(t)}_{\text{decaying terms}}$$

$$x(t) = x_h(t) + x_p(t)$$

$x_p(t) \to A_m \cos(\omega t + \theta + \phi)$

as $t \to \infty$, $\boxed{x(t) = x_{h_0}(t) + \underbrace{A_m \cos(\omega t + \theta + \phi)}_{x_p(t)}}$

$x_p(t) \to$ SSS (sinusoidal steady state)

SS $\to$ steady state part

b) $s_k = j\omega$ ⟶ unbounded situation

$$X_p(t) = t^{d_k} A_m \cos(\omega t + \theta + \phi)$$

(4) $Re\{s_i\} > 0$ for some s_i (not important to look at the input)

$x(t)$ is unstable

(5) $Re\{s_i\} = 0$ and $s_i = j\omega_0$ but s_i has a multiplicity, $X(t)$ ⟶ unbounded

$j\omega_0$: different from input angular frequency ω ($\omega_0 \neq \omega$)

unbounded solution ⟶ no steady st

Case 2:

$$D = \frac{d}{dt}$$

$$D' = \frac{d}{dt}$$

$$D^2 = \frac{d^2}{dt^2}$$

$$(D^3 + 4D^2 + 6D + 4)\, X(t) = u(t)$$

char - equation $= s^3 + 4s^2 + 6s + 4 = 0$

$$(s+1+j)(s+1-j)(s+2) = 0$$

$s_1 = -1-j$

$s_2 = -1+j$ $\longrightarrow$ all multiplicities are 1.

$s_3 = -2$

$Re\{s_1\} < 0 \quad Re\{s_2\} < 0 \quad Re\{s_3\} < 0$

2^{nd} cased

$$X_h(t) = \underbrace{A_1 e^{-2t}}_{\text{due to } s_3=-2} + \underbrace{A_2 e^{-t}\cos(t) + A_3 e^{-t}\sin(t)}_{\text{due to } s_1 = s_2^*}$$

$s_1 = -1+j \quad s_2 = -1-j$

assume $u(t) = 4\cos(2t + 30°)$

$$X_p(t) = K\cos(2t + \varnothing) = K_1\cos(2t) + K_2\sin(2t)$$

put this function in diff- eqn. and find K_1 and K_2

$$X_p(t) = \frac{1}{\sqrt{10}}\cos\left(2t + 30 \underbrace{- 180° + \tan^{-1}\left(\frac{1}{3}\right)}_{?}\right)$$

$$X(t) = X_h(t) + X_p(t)$$

$$\lim_{t \to \infty} X(t) = \underbrace{X_p(t)}_{\text{bounded}} \quad SSS$$

Case _ 3a

Ex: $(D^3+D^2+D+1)\,X(t)=u(t)=4\cos(2t+30°)$

$\omega = 2$

$s_k \mp j2$

$s^3+s^2+s+1=0=(s+1)(s+j)(s-j)=0$

$s_1=-1 \qquad Re\{s_1\}<0$

$s_2=-j \qquad Re\{s_2\}=0 \qquad$ Case 3a is provi

$s_3=j \qquad Re\{s_3\}=0$

$X_h(t)=K_1e^{-t}+K_2\cos t+K_3\sin t$ ($\mp j$)

$$X_p(t)=\frac{4}{\sqrt{45}}\cos\Big(2t+30°+\underbrace{180°-\tan^{-1}(2)}_{?}\Big)$$

$X(t)=X_h(t)+X_p(t)$

as $t\longrightarrow\infty \qquad X(t)=\underbrace{K_2\cos(t)+K_3\sin(t)+\underbrace{X_p(t)}_{SSS}}_{SS}$

Case 3b:

$(D^3+D^2+4D+4)\,X(t)=u(t)=4\cos(2t+30°)$

$s^3+s^2+4s+4=0$

$(s+2j)(s-2j)(s+1)=0$

$s_1=2j \qquad s_2=-2j \qquad s_3=-1$

$Re\{s_1\} = Re\{s_2\} = 0 \quad Re\{s_3\} < 0 \quad \boxed{\omega = 2}$

$\boxed{s_1 = 2j = j\omega} \longrightarrow$ case 3b

$X_h(t) = K_1 e^{-t} + K_2 \cos(2t) + K_3 \sin(2t)$

$X_p(t) = \underline{Mt} \cos \underline{(2t + \emptyset)}$

$X_p(t) = t\left[M_1 \cos 2t + M_2 \sin 2t\right]$

$X_p(t) = \frac{1}{\sqrt{5}} \cancel{\bullet} X \cos\left(2t + 30 - \underbrace{180^\circ + \tan^{-1}\left(\frac{1}{2}\right)}\right)$

?

$X(t) = X_h(t) + X_p(t)$

as $t \longrightarrow \infty \quad X(t) = K_2 \cos(2t) + K_3 \sin(2t) + \underbrace{X_p(t)}$

$X(t) \longrightarrow \mp \infty$

unbounded

no steady-state solution exist

4^{th} case:

Ex: $(s-1)(s+1)(s+j)(s-j) = 0$

$u(t) = 4 \cos(2t + 30^\circ)$

$s_1 = 1 \quad s_3 = -j \quad K_1 e^{t} + K_2 e^{-t} + K_3 \cos(t) + K_4 \sin(t)$

$s_2 = -1 \quad s_4 = j$

$$X_h(t) = K_1 e^{t} + K_2 e^{-t} + K_3 \cos t + K_4 \sin t$$

$$X_p(t) = M \left[\cos(2t + \phi) \right]$$

as $t \longrightarrow \infty$; $X(t) \longrightarrow \mp \infty$

Case 5:

$$(D^5 + D^4 + 2D^3 + 2D^2 + D + 1)\, X(t) = u(t) = 4 \cos(2t + 30$$

$$(s+1)\,(s+j)^2\,(s-j)^2 = 0$$

$s_1 = -1$ $s_2 = -j$ $s_3 = j$
multp $= 2$ multp $= 2$

$$X_h(t) = K_1 e^{-t} + K_2 \cos(t) + K_3 \sin(t) + \underbrace{K_4\, t \cos t}$$
$$\underbrace{+ K_5\, t \sin t}$$

due to multiplicity
s_2 and s_3

$s_k \neq jw = 2j$ $k = 1 ; 2 ; 3$

$$X_p(t) = M \cos(2t + \phi)$$

$$= \frac{4}{\sqrt{2197}} \cos\left(2t + 30^\circ - \tan^{-1}\left(\frac{46}{9}\right)\right)$$